RAMANUJAN-SUM EXPANSIONS FOR FINITE DURATION (FIR) SEQUENCES

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ABSTRACT

Ramanujan sums have in the past been used to represent arithmetic sequences. It is shown here that for finite duration (FIR) sequences with length N, the traditional representation is not suitable. Two new types of Ramanujan-sum expansions are proposed here for the FIR case, each offering an *integer basis.* One of these is particularly suited to identify periodicities in the FIR sequence. This representation in fact expresses any FIR sequence as a sum of orthogonal sequences each with a hidden periodicity corresponding to a divisor of N.

Index Terms— Ramanujan sums, periodicity, periodic subspaces, integer basis, periodic orthogonal projections.

1. INTRODUCTION

In 1918 the famous Indian mathematician Ramanujan introduced a trigonometric summation, now called the Ramanujan sum. Given an integer q, the qth Ramanujan sum (RS) is defined as [11]

$$c_q(n) = \sum_{\substack{k=1\\(k,q)=1}}^{q} e^{j2\pi kn/q} = \sum_{\substack{k=1\\(k,q)=1}}^{q} W_q^{-kn} \tag{1}$$

where $W_q = e^{-j2\pi/q}$. Here the notation (k,q) denotes the greatest common divisor (gcd) of k and q. Thus $(k,q) = 1$ means k and q are coprime. For example $c_9(n) = W_9^{-n} + W_9^{-2n} + W_9^{-4n} + W_9^{-5n} + W_9^{-7n} + W_9^{-8n}$. Comparing (1) with the inverse DFT formula, it follows that the q-point DFT of $c_q(n)$ is $C_q[k] = q$ if $(k,q) = 1$ and zero otherwise. This can be regarded as an equivalent definition of $c_q(n)$. Ramanujan's motivation in introducing this sum was to show that several standard arithmetic functions in the theory of numbers can be expressed as linear combinations of $c_q(n)$, that is,

$$x(n) = \sum_{q=1}^{\infty} \alpha_q c_q(n), \quad n \geq 1, \tag{2}$$

An arithmetic function is a sequence (i.e., function of integer argument), and is usually (but not necessarily) integer valued. Examples include the Möbius function $\mu(n)$, Euler's totient function $\phi(n)$, the von Mangoldt function $\Lambda(n)$, and the Riemann-zeta function $\zeta(s)$ [6]. The Ramanujan expansion (2) was derived in [11] for many arithmetic functions. Eq. (2) is sometimes referred to as the Ramanujan Fourier transform expansion (i.e., α_q are the RFT coefficients) [10]. A number of authors have used the following formula for the calculation of the coefficients α_q [2], [4], [9]:

$$\alpha_q = \frac{1}{\phi(q)} \left(\lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} x(n) c_q(n) \right) \tag{3}$$

Here, the **Euler totient** function $\phi(q)$ is the number of integers in $1 \leq k \leq q$ coprime to q. The application of Ramanujan sums in the context of signal processing has also been examined by a number of authors [9], [10], [3], [12], [8], [14], [7]. Applications in the representation of certain periodic signals was demonstrated by Planat (e.g., see [10]). Time-frequency analysis of signals based on Ramanujan expansions was considered in a letter by Sugavaneswaran [14], based on the 2D version of RS. An application in cardiology was described in [7]. A very insightful connection between Ramanujan sums and the proof of the famous twin-prime conjecture was established by Gadiyar and Padma in [4] based on the possibility of a Wiener-Kintchine like formula for Ramanujan expansions. The conjecture itself has been proved recently by Yitang Zhang [17].

In this paper we consider finite duration (FIR) sequences $x(n)$ of arbitrary length N. We show in Sec. 3 that the expansion formula based on (3) will not work, and propose a simple solution. Then in Sec. 4 we introduce a novel way to represent arbitrary FIR sequences using Ramanujan sums. This representation expresses $x(n)$ as a sum of orthogonal sequences each with a hidden periodicity corresponding to a divisor of N. Since $c_q(n)$ are integers, this offers an integer basis for FIR sequences. We also demonstrate the application of this representation in identifying periodic components. For convenience a brief review of basic properties of Ramanujan sums is included in Sec. 2.

2. BASIC PROPERTIES OF RAMANUJAN SUMS

From the definiton (1) we see that $c_q(n) = c_q(n+q)$ for all n, so $c_q(n)$ is periodic. Observe also that $c_q(n)$ is always real. This is because if $(k,q) = 1$ then $(q-k,q) = 1$, so $e^{j2\pi kn/q}$ and $e^{-j2\pi kn/q}$ both appear in the sum (1). It therefore also follows that $c_q(n) = \sum_{(k,q)=1} \cos(2\pi kn/q)$ (Ramanujan's original definition [11]), which immediately implies $c_q(n) = c_q(-n)$. Thus $c_q(n)$ is *real, periodic, and symmetric.* It can be written in either one of the forms

$$c_q(n) = \sum_{\substack{k=1\\(k,q)=1}}^{q} W_q^{kn} = \sum_{\substack{k=1\\(k,q)=1}}^{q} W_q^{-kn} \tag{4}$$

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where $W_q = e^{-j2\pi/q}$. What is less obvious is the fact that $c_q(n)$ is **integer valued** for all q and n. Ramanujan proved this by proving the relation $c_q(n) = \sum_{d|(q,n)} d\mu(q/d)$ where $\mu(n)$ is the Möbius function [6], and the sum is over all integers d which are divisors of q and n. Since $\mu(n)$ is an integer for all n, it follows that $c_q(n)$ is an integer for all n and q. Here are the first few Ramanujan sums written for one period:

$$c_1(n) = 1,\ c_2(n) = \{1, -1\},\ c_3(n) = \{2, -1, -1\},$$
$$c_4(n) = \{2, 0, -2, 0\},\ c_5(n) = \{4, -1, -1, -1, -1\}\ \ldots$$

The following orthogonality property of Ramanujan sequences is crucial (here $m = \mathrm{lcm}(q_1, q_2)$, where lcm denotes *least common multiple*):

$$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n-l) = 0, q_1 \neq q_2 \tag{5}$$

for any integer l. This can be proved using the result $\sum_{n=0}^{m-1} W_{q_1}^{-k_1 n} W_{q_2}^{k_2 n} = 0$, when $(k_1, q_1) = (k_2, q_2) = 1$ and $1 \leq k_i \leq q_i$.

3. FIR SEQUENCES AND RAMANUJAN SUMS

Arithmetic functions (for which Ramanujan expansions were originally used) are infinite duration sequences, and the coefficients have to be evaluated through the limiting process (3). If $x(n)$ is FIR with support $1 \leq n \leq N$, then

$$\lim_{M\to\infty} \sum_{n=1}^{M} x(n)c_q(n)/M = \lim_{M\to\infty} \sum_{n=1}^{N} x(n)c_q(n)/M \to 0$$

which shows that $\alpha_q \to 0$ for each q. Thus, the conventional approach does not lead to a correct expansion of the form (2). So let us try a different approach. Given an FIR $x(n)$, let us pretend that it is one period of a periodic signal, so that

$$x(n) = x(n+N) \tag{6}$$

For fixed q if the limit in (3) exists then in particular it is equal to $\lim_{k\to\infty} \sum_{n=1}^{kqN} x(n)c_q(n)/kqN$. But $x(n)$ and $c_q(n)$ have periods N and q respectively, so $x(n)c_q(n)$ repeats every qN samples. So the above limit equals $\sum_{n=1}^{qN} x(n)c_q(n)/qN$, and

$$\alpha_q = \frac{1}{qN\phi(q)} \sum_{n=1}^{qN} x(n)c_q(n) \tag{7}$$

Using $x(n) = x(n+N)$ and the expression (4) for $c_q(n)$ we can rewrite

$$\begin{aligned}\sum_{n=1}^{qN} x(n)c_q(n) &= \sum_{n=1}^{N} x(n) \sum_{i=0}^{q-1} c_q(n+iN) \\ &= \sum_{n=1}^{N} x(n) \sum_{\substack{k=1\\(k,q)=1}}^{q} W_q^{kn} \sum_{i=0}^{q-1} W_q^{ikN}\end{aligned}$$

Since $(k, q) = 1$, the inner sum is zero unless N is a multiple of q. Thus $\alpha_q = \sum_{n=1}^{N} x(n)c_q(n)/N\phi(q)$ when q is a divisor of N, and is zero otherwise. So the representation of a length N FIR sequence $x(n)$ is given by

$$x(n) = \sum_{q_i|N} \alpha_{q_i} c_{q_i}(n), \quad 1 \leq n \leq N, \tag{8}$$

where $q_i|N$ means that q_i is a divisor of N, and where

$$\alpha_{q_i} = \frac{1}{N\phi(q_i)} \sum_{n=1}^{N} x(n) c_{q_i}(n) \tag{9}$$

The number of terms in (8) is equal to the number of divisors of N, which is *less* than N (unless $N = 1$). For example, if $N = 6$ its divisors are $\{1, 2, 3, 6\}$, so (8) reduces to $x(n) = \alpha_1 c_1(n) + \alpha_2 c_2(n) + \alpha_3 c_3(n) + \alpha_6 c_6(n)$. Thus the number of free coefficients α_{q_i} is less than the number of samples in $x(n)$ which shows that *an arbitrary FIR sequence cannot be represented* as in (2) (i.e., as in (8)) if α_q is as in (3).

The good news however is that we *can* successfully represent an FIR sequence using Ramanujan sums if we change our approach a little bit. Thus consider the expansion

$$x(n) = \sum_{q=1}^{N} a_q c_q(n), \quad 0 \leq n \leq N-1 \tag{10}$$

where the first N sequences $c_q(n)$ are used. For convenience the support is assumed to be $0 \leq n \leq N-1$. In matrix vector form

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\mathbf{x}} = \mathbf{A}_N \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}}_{\mathbf{a}} \tag{11}$$

where the qth column of $\mathbf{A}_N$ has the elements $c_q(n)$ repeated with period q until we get N rows. For example,

$$\mathbf{A}_6 = \begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 2 \\ 1 & -1 & -1 & 0 & -1 & 1 \\ 1 & 1 & -1 & -2 & -1 & -1 \\ 1 & -1 & 2 & 0 & -1 & -2 \\ 1 & 1 & -1 & 2 & -1 & -1 \\ 1 & -1 & -1 & 0 & 4 & 1 \end{bmatrix} \tag{12}$$

The Ramanujan expansion (10) will always be successful because the $N \times N$ matrix $\mathbf{A}_N$ always has full rank [16]. An outline of the proof is as follows: by using elementary column operations the preceding matrix can be transformed into a triangular matrix with diagonal elements $1, 2, \ldots, q$, so that $|\det \mathbf{A}_q| = q! \neq 0$. The preceding transformation is justified in [16] based on the recursion

$$c_q(n) = q\delta((n))_q - \sum_{\substack{q_i|q\\q_i<q}} c_{q_i}(n), \tag{13}$$

which is proved in [16]. Here $\delta((n))_q = 1$ when n is a multiple of q, and zero otherwise.

The columns of $\mathbf{A}_N$ form an integer basis for $\mathbb{C}^N$. But notice that many of the columns do *not* represent an integer number of periods of $c_q(n)$. So the columns of $\mathbf{A}_N$ are *not* orthogonal. In the next section we shall introduce another way to represent an FIR sequence as a finite sum of Ramanujan sums, which is more elegant in some ways. Notice incidentally that the recursion (13) yields an inductive proof, without requiring the use of Möbius functions, that $c_q(n)$ is an integer (the basis of induction being that $c_1(n)$ is obviously an integer).

4. A NEW RAMANUJAN REPRESENTATION FOR FIR SEQUENCES

The representation (8) was not successful because the number of nonzero α's is less than N. In matrix form Eq. (8) becomes

$$\mathbf{x} = \begin{bmatrix} \mathbf{c}_{q_1} & \mathbf{c}_{q_2} & \dots & \mathbf{c}_{q_K} \end{bmatrix} \mathbf{d} \tag{14}$$

where K is the number of divisors of N. The column $\mathbf{c}_{q_i}$ has the elements $c_{q_i}(n)$ repeated with period q_i until there are N rows. Since q_i is a divisor of N, there are an integer number (N/q_i) of periods in $\mathbf{c}_{q_i}$. The matrix has a column space with dimension $K < N$, and only those FIR sequences which are in this column space can be represented. Now instead of using only one column for each q_i, consider using $\phi(q_i)$ *circularly shifted versions*, i.e., for each q_i define the matrix

$$\mathbf{G}_{q_i} = \begin{bmatrix} \mathbf{c}_{q_i} & \mathbf{c}_{q_i}^{(1)} & \dots & \mathbf{c}_{q_i}^{(\phi(q_i)-1)} \end{bmatrix} \tag{15}$$

where $\mathbf{c}_{q_i}^{(k)}$ represents circular downshifting by k. We will show below that this matrix has rank $\phi(q_i)$. Next define the composite matrix

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{G}_{q_1} & \mathbf{G}_{q_2} & \dots & \mathbf{G}_{q_K} \end{bmatrix} \tag{16}$$

This has N rows and $\sum_{q_i|N} \phi(q_i)$ columns. But it is well-known (p. 65, [6]) that

$$\sum_{q_i|N} \phi(q_i) = N \tag{17}$$

That is, the sum of the Euler totients, taken over all the divisors of N, is precisely equal to N. Thus the matrix $\mathbf{F}_N$ is $N \times N$ and can serve as a basis for any FIR sequence $\mathbf{x}$ provided it has full rank. Before discussing the rank issue, let us look at an example. Let $N = 6$. Then the divisors are $q_1 = 1,\ q_2 = 2,\ q_3 = 3,\ q_4 = 6$. So the matrix in (14) is

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{18}$$

which is only 6×4. But since $\phi(q_1) = 1, \phi(q_2) = 1, \phi(q_3) = 2, \phi(q_4) = 2$, we have

$$\mathbf{F}_6 = \begin{bmatrix} 1 & 1 & 2 & -1 & 2 & 1 \\ 1 & -1 & -1 & 2 & 1 & 2 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 2 & -1 & -2 & -1 \\ 1 & 1 & -1 & 2 & -1 & -2 \\ 1 & -1 & -1 & -1 & 1 & -1 \end{bmatrix} \tag{19}$$

which is a 6×6 matrix indeed. We now prove the following:

Theorem 1. The matrix $\mathbf{F}_N$ has full rank N. $\diamondsuit$

Proof. Since $q_i|N$, each column in (16) has an integer number of periods of $c_{q_i}(n)$. So, in view of the orthogonality property (5), the column space of $\mathbf{G}_{q_i}$ is orthogonal to that of $\mathbf{G}_{q_k}$ for $i \neq k$. So it only remains to prove that the $\phi(q_i)$ columns in each $\mathbf{G}_{q_i}$ are linearly independent. With $\mathbf{e}_q$ denoting the vector with the q samples of $c_q(n)$, and $\mathbf{w}_q^{(k)}$ denoting the kth column of the $q \times q$ DFT matrix, we can write $\mathbf{e}_q = \sum_i \mathbf{w}_q^{(k_i)}$ where $(k_i, q) = 1$ and $1 \leq k_i \leq q$. Thus

$$\begin{aligned} &\begin{bmatrix} \mathbf{e}_q & \mathbf{e}_q^{(1)} & \dots & \mathbf{e}_q^{(\phi(q)-1)} \end{bmatrix} \\ = &\underbrace{\begin{bmatrix} \mathbf{w}_q^{(k_1)} & \mathbf{w}_q^{(k_2)} & \dots & \mathbf{w}_q^{(k_{\phi(q)})} \end{bmatrix}}_{\mathbf{W}_1} \mathbf{U} \end{aligned} \tag{20}$$

where $\mathbf{U}$ is a $\phi(q) \times \phi(q)$ submatrix of the $q \times q$ DFT matrix, obtained by retaining the first $\phi(q)$ columns, and the $\phi(q)$ rows whose indices are coprime to q. Since $\mathbf{U}$ has Vandermonde rows $\begin{bmatrix} 1 & W^{k_i} & W^{2k_i} & \dots \end{bmatrix}$, and since W^{k_i} are distinct for different i, $\mathbf{U}$ has rank $\phi(q)$. And since $\mathbf{W}_1$ is a submatrix of the DFT with $\phi(q)$ columns it has rank $\phi(q)$. So the product in Eq. (20) has rank $\phi(q)$. So we have proved that in Eq. (16) the first $\phi(q_i)$ rows of each $\mathbf{G}_{q_i}$ are linearly independent. So each $\mathbf{G}_{q_i}$ has rank $\phi(q_i)$ indeed. $\bigtriangledown\bigtriangledown\bigtriangledown$

We have therefore shown that any $N \times 1$ vector $\mathbf{x}$ can be represented in the form

$$\mathbf{x} = \mathbf{F}_N \mathbf{b} \tag{21}$$

where the $N \times N$ matrix $\mathbf{F}_N$ is defined in terms of the Ramanujan sums c_{q_i} as described above. Thus $\mathbf{F}_N$ defines an integer basis for $\mathbb{C}^N$. Summarizing, we have proved:

Theorem 2. Any length N sequence can be represented as a linear combination of the form

$$x(n) = \sum_{q_i|N} \underbrace{\sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l)}_{x_{q_i}(n)} \tag{22}$$

where $c_{q_i}(n)$ are Ramanujan sums. $\diamondsuit$

By (17), the total number of terms in the double summation (22) is precisely N. The inner sum $x_{q_i}(n)$ defines a subspace spanned by the $\phi(q_i)$ shifted versions $c_{q_i}(n-l)$. It will be called the q_ith **Ramanujan subspace** $\mathcal{S}_{q_i}$. This is nothing but the subspace spanned by the $\phi(q_i)$ columns of $\mathbf{G}_{q_i}$. Since the column spaces of $\mathbf{G}_{q_i}$ are orthogonal for different q_i, the components $x_{q_i}(n)$ are *orthogonal projections* of $x(n)$ onto the K Ramanujan subspaces. So, the theorem says that *any* FIR $x(n)$ can be decomposed into a sum of K orthogonal Ramanujan projections where each projection $x_{q_i}(n)$ is a linear combination of $\phi(q_i)$ uniformly shifted versions of the integer sequences $c_{q_i}(n)$.

It can be shown that the DFT expansion of $x(n)$ can be rearranged as $x(n) = \sum_{q_i|N} \sum_{(k,q_i)=1} \gamma_{q_i,k} W_{q_i}^{kn}$, which is similar to (22). However, unlike the DFT, the Ramanujan matrices $\mathbf{A}_N$ and $\mathbf{F}_N$ have real integer coefficients, so all the computations can be performed by addition and subtraction of the samples in $x(n)$. Since there are $\phi(q)$ terms in (4), $c_q(n)$ are "reasonably small," i.e., $|c_q(n)| \leq \phi(q) < q$. Notice also that N is not restricted as in other integer transforms such as the Hadamard transform.

An application. One past motivation [10] for using the original Ramanujan representation (Eqs. (2) and (3)) was that it tended to highlight the periodicities in a signal. However, as we showed in Sec. 3, the use of (3) cannot yield a correct representation for arbitrary FIR sequences. But if we use the representation in (22), we can prove the following results [16]: (a) The orthogonal projection $x_{q_i}(n)$ has period exactly equal to q_i. (b) Among all the projections $x_{q_i}(n)$ suppose the nonzero ones are $x_{q_{i_k}}(n)$ for $1 \leq k \leq K_z$ $(K_z \leq K)$. Then $x(n)$ has periodicity exactly equal to the lcm of these q_{i_k}. Thus the representation allows us to identify the exact periodicity, by identifying the integers q_{i_k} corresponding to nonzero projections. More importantly, it is shown in [16] that the *projections $x_{q_i}(n)$ can themselves be computed by integer operations* (add and subtract operations) on (the possibly complex) $x(n)$. Eq. (22) performs a periodicity transform [13]. We call it the *Ramanujan Periodicity Transform* or **RPT**.

Example 1. Fig. 1(a) shows an FIR signal $x(n)$ with $N = 2^8$ points. Within its support, $x(n)$ is periodic with period $2^6 = 64$ (a divisor of N). Figure 1(b) shows the Ramanujan coefficients a_k (absolute values) in the expansion (11). The plot does not reveal anything about the period 64. Next consider the new representation (22) (equivalently (21)). Since $N = 2^8$, the divisors are all the powers of 2 up to 2^8. So the 9 Ramanujan subspaces are $\mathcal{S}_{2^k}$ where $0 \leq k \leq 8$. Fig. 1(c) shows the coefficients b_k (absolute values) in the expansion (21) (or equivalently β_{il} in appropriate order), and Fig. 1(d) shows the energies of the projections x_{q_i} in (22). The projection are zero for $\mathcal{S}_{2^8}$ and $\mathcal{S}_{2^7}$. So the only nonzero projections are $x_{2^k}, 0 \leq k \leq 6$. Since the lcm of these divisors with nonzero projections is 2^6, it follows that the period is $2^6 = 64$ indeed. So, the new Ramanujan decomposition can be used to identify periodic components in FIR sequences. The DFT of $x(n)$ (not shown) can in principle reveal that $x(n)$ has period 64 (since $X[k]$ is nonzero only when k is a multiple of 4 $(= N/64)$), but the plots in Fig. 1(c), (d) are more direct, and can be obtained with integer transforms.

5. CONCLUDING REMARKS

The traditional way of expanding a signal using Ramanujan sums does not work in the FIR case. In this paper we developed a new way to do this decomposition. Using this, any length-N signal $x(n)$ can be decomposed as a sum of orthogonal projections into spaces $\mathcal{S}_{q_i}$ each representing a periodic component x_{q_i} (where q_i are divisors of N). Many issues remain to be analyzed such as complexity, the case where periods are not divisors of N, and the presence of noise. Connecttions to dictionary based approaches (see [15] and references therein) will be explored in future.

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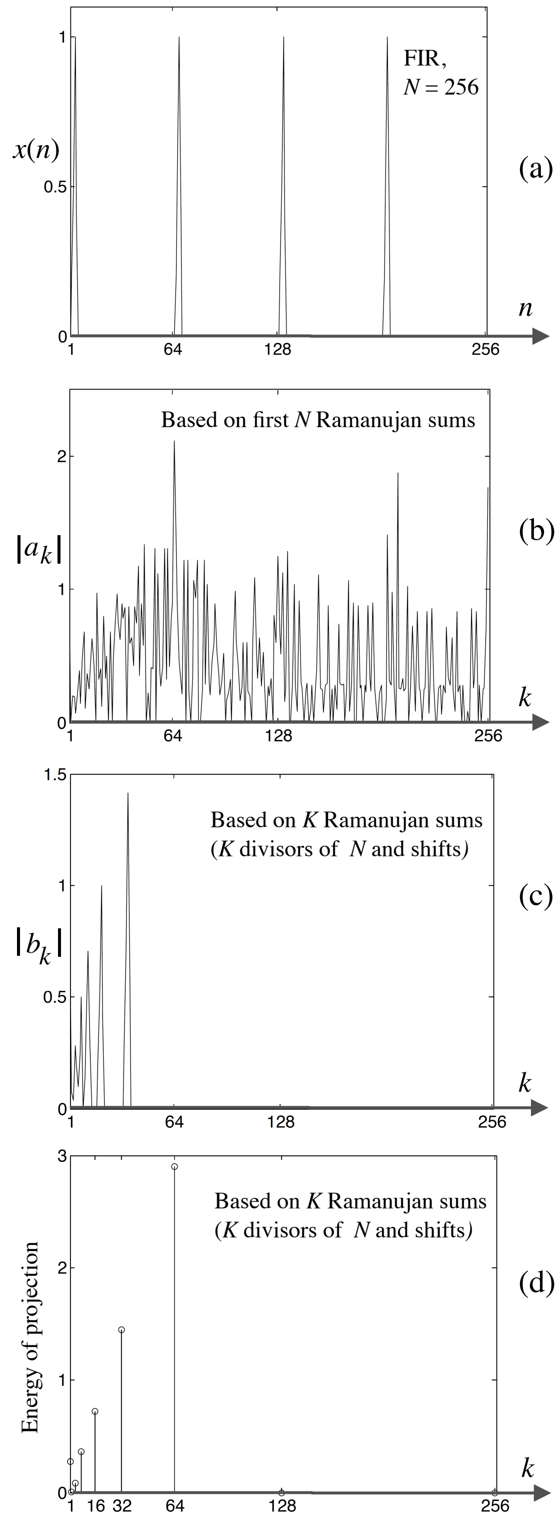

Fig. 1. Example 1. (a) The 256-point FIR signal $x(n)$ with period 64. (b) The Ramanujan coefficients a_k (absolute values) corresponding to the basis $\mathbf{A}_N$. (c) The Ramanujan coefficients b_k (absolute values) corresponding to the basis $\mathbf{F}_N$ (the representation (22)). (d) Energies of the orthogonal projections x_{q_k} into the Ramanujan subspaces $\mathcal{S}_{q_k}$.

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