Summary and Generalization

\[ y' + ay = x \rightarrow H(s) = \frac{1}{s + a}. \]

\[ \sum_{j=0}^{N} a_j \frac{d^j}{dt^j} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dy^k} x(t) \]

\[ H(s) \sum_{j} A(s) a_j s^j = \sum_{k} B(s) b_k s^k. \]
\[ H(s) = \frac{B(s)}{A(s)} \quad \text{(Rational function of } s) \]
Discrete Time is Similar

\[ y[n - 1] + ay[n] = x[n] \rightarrow H(z) = \frac{1}{z^{-1} + a}. \]

\[
\sum_{j=0}^{N} a_j y[n - j] = \sum_{k=0}^{M} b_k x[n - k]
\]

\[
H(z) \sum_{j} A(z)^j = \sum_{k} B(z)^k.
\]

\[
H(z) = \frac{B(z)}{A(z)} \quad \text{(Rational function of } z^{-1})
\]
Review of Continuous-Time LTI Systems

Continuous time:

\[
\begin{align*}
\delta(t) &\mapsto h(t) \\
x(t) &\mapsto x(t) \ast h(t) = \int x(\tau)h(t - \tau)d\tau \\
e^{st} &\mapsto H(s)e^{st} \\
X(s) &\mapsto X(s)H(s) \\
H(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt \quad \text{(Laplace Transform)}
\end{align*}
\]

But where (i.e., for which values of \(s\)) does \(H(s)\) converge? In particular, does the frequency response \(H(j\omega)\) exist?
Review of Discrete-Time LTI Systems

\[ \begin{align*}
\delta[n] & \mapsto h[n] \\
 x[n] & \mapsto x[n] * h[n] = \sum_{k} x[k]h[n - k] \\
 z^n & \mapsto H(z)z^n \\
 X(z) & \mapsto X(z)H(z) \\
 H(z) & = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad \text{(z-transform)}
\end{align*} \]

But where (i.e., for which value \( z \)) does \( H(s) \) resp. \( H(z) \) converge? In particular, does the frequency response \( H(e^{j\omega}) \) exist?
Example A

Here are two discrete-time signals with the same $z$-transform (but different ROCs.)

\[
x_1[n] = u[n]
\]
\[
x_2[n] = -u[-n - 1]
\]

\[
X_1(z) = \sum_{n \geq 0} z^{-n} = \frac{1}{1 - z^{-1}} \quad \text{if } |z| > 1
\]

\[
X_2(z) = - \sum_{n \geq 1} z^n = -\frac{z}{1 - z} \quad \text{if } |z| < 1
\]

Note that neither series converges anywhere on the circle \{|z| = 1\}, since the terms of the series don’t approach 0. Still, the function $z/(z - 1)$ is analytic everywhere except $z = 1$. 
Example B

Here are two continuous-time signals with identical Laplace transforms (but different ROC’s).

\[ x_1(t) = u(t) \]
\[ x_2(t) = -u(-t) \]

\[ X_1(s) = \int_{0}^{\infty} e^{-st} dt = s^{-1} \quad \text{Re } s > 0 \]

\[ X_2(s) = -\int_{-\infty}^{0} e^{-st} dt = s^{-1} \quad \text{if Re } s < 0 \]
The Z-Transform is Easier

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

**Theorem 1.** Define real numbers \( r_R \) and \( r_L \) as follows:

1. \( r_R = \limsup_{n \to \infty} |x[n]|^{1/n} \)
2. \( r_L = \liminf_{n \to \infty} |x[-n]|^{-1/n} \)
3. \( = \liminf_{n \to -\infty} |x[n]|^{1/n} \).
Statement of Theorem

**Theorem.** If \( r_R < |z| < r_L \), then \( X(z) \) converges absolutely. On the other hand if \( |z| > r_L \) or if \( |z| < r_R \), then \( X(z) \) diverges. If \( |z| = r_L \) or \( |z| = r_R \), no general conclusion is possible.
Corollary. If the signal is right-sided ($x[n] = 0$ for $n \leq n_0$), then $r_L = \infty$, and the ROC is $\{z : |z| > r_R\}$. On the other hand, if the signal is left-sided, i.e., $x[n] = 0$ for $n \geq n_0$, then $x_R = 0$ and the ROC is $\{z : |z| < r_L\}$. 
By definition, the Laurent series

\[ X(z) = \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \]

converges iff the power series

\[
\begin{align*}
X_L(z) &= \sum_{n>0} x[-n]z^n \\
X_R(z) &= \sum_{n\geq0} x[n]z^{-n}
\end{align*}
\]

both converge
A Famous Theorem from Complex variables

Theorem 2. A power series of the form

\[ f(z) = \sum_{n \geq 0} a_n z^n \]

(where the \( a_n \)'s are complex numbers) converges absolutely for \( |z| < r \) and diverges for \( |z| > r \), where the nonnegative number \( r \) (the radius of convergence) is given by the formula

\[ r = \liminf_{n \to \infty} |a_n|^{-1/n}. \]

Furthermore, the function \( f(z) \) must have one or more singularities on the critical circle \( \{ z : |z| = r \} \). \( \blacksquare \)
For the Record

If $f(z)$ is analytic inside $C$,

$$a_n = \frac{1}{2\pi j} \oint_C \frac{f(z)}{z^{n+1}} \, dz.$$
Example 1

Let the sequence $x[n]$ be defined as follows:

$$x[n] = \begin{cases} 
2^{-n} & \text{if } n \geq 0 \text{ and } n \text{ is even} \\
3^{-n} & \text{if } n \geq 0 \text{ and } n \text{ is odd} \\
2^n & \text{if } n < 0.
\end{cases}$$

For $n \geq 0$ we have $|x[n]|^{1/n} = 1/2$ if $n$ is even, and $= 1/3$ if $n$ is odd. Thus from (1) we find that $r_R = 1/2$. For $n < 0$ we have $|x[n]|^{1/n} = 2$, so that by (2) we have $r_L = 2$. Thus the ROC for this example is the region \$\{1/2 < |z| < 2\}$.

In this case $H(e^{j\omega})$ exists. •
Example 2

Now let

\[ x[n] = \begin{cases} 
1 & \text{if } n \geq 0 \\
0 & \text{if } n \leq 0. 
\end{cases} \]

Then from (1), we get \( r_R = 1 \), and from (2) we get \( r_L = \infty \). Thus the ROC for this signal is \( \{ z : |z| > 1 \} \). Note that in this case \( X(z) \) converges nowhere on the critical circle \( \{ |z| = 1 \} \), since the terms of the series \( X(z) \) do not approach zero for \( |z| = 1 \).

In short, \( H(e^{j\omega}) \) doesn’t exist. ■
Example 3

Consider next

\[ x[n] = \begin{cases} 
  1/n & \text{if } n \geq 1 \\
  0 & \text{if } n \leq 0. 
\end{cases} \]

Once again from Theorem 1, we get \( r_R = 1 \), and \( r_L = \infty \). Thus the ROC for this signal is also \( \{z : |z| > 1\} \). In this case \( X(z) \) diverges for \( z = 1 \), and (I think) converges for all other values of \( z \) on the critical circle \( |z| = 1 \). \( \blacksquare \)
Finally, consider

\[ x[n] = \begin{cases} 
1/n^2 & \text{if } n \geq 1 \\
0 & \text{if } n \leq 0.
\end{cases} \]

Once again Theorem 1 tells us that \( r_R = 1 \), and \( r_L = \infty \). The interesting thing here is that \( X(z) \) converges absolutely everywhere on the critical circle \( |z| = 1 \), since the series \( \sum_{n \geq 1} n^{-2} \) converges. \( \blacksquare \)
Stability

**Theorem 3.** If the open ROC for $H(z)$ contains the point $z = 1$, then the system is stable. If the point $z = 1$ lies outside the closed ROC, then the system is unstable. On the other hand, if $z = 1$ lies on one of the critical circles, no general conclusion is possible.

**Corollary.** If $h[n]$ is causal, then the system is stable iff $H(z)$ has no singularities with $|z| \geq 1$.

**Corollary.** If $h[n]$ is anticausal, then the system is stable iff $H(z)$ has no singularities with $|z| < 1$. 
Relationship to Fourier Transform

\[ X_L(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \]

\[ X_{\mathcal{F}}(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega}dt \]

\[ X_{\mathcal{F}}(j\omega) = X_L(s)|_{s=j\omega} \]

\[ X_L(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \]

\[ = \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\omega} dt \]

\[ = X_{\mathcal{F}}(x(t)e^{-\sigma t}) \]
Relationship to z-Transform

\[ X_{\mathcal{F}}(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n} \]

\[ X_{\text{zee}}(z) = \sum_{-\infty}^{\infty} x[n]z^{-n} \]

\[ X_{\mathcal{F}}(e^{j\omega}) = X_{\text{zee}}(z) \bigg|_{z=e^{-j\omega}}. \]

\[ X_{\text{zee}}(re^{j\omega}) = \sum_{-\infty}^{\infty} x[n](r^{-n}e^{-j\omega n}) \]

\[ = \sum_{-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n} \]

\[ = X_{\mathcal{F}}(x[n]r^{-n}) \]
Unilateral Laplace Transforms

\[ X(s) = \int_0^\infty x(t)e^{-st}dt. \]

Suppose \( X(s) \) converges for some real number \( a \). Then \( X(s) \) converges for all \( s \) with \( \Re s > a \). The number

\[ \sigma = \inf\{a : X(a) \text{ converges}\} \]

is called the abscissa of convergence of \( X(s) \). Computationally,

\[ \sigma = \lim_{t \to \infty} \frac{1}{t} \log |x(t)|. \]

if the limit exists.
Bilateral Laplace Transforms

\[ \int_{\infty}^{-\infty} x(t) e^{-st} dt = \int_{\infty}^{0} x(t) e^{-st} dt + \int_{-\infty}^{\infty} x(t) e^{-st} dt \]
\[ = \int_{0}^{\infty} x(t) e^{-st} dt + \int_{0}^{\infty} x(-u) e^{su} du. \]

\[ X(s) = X_R(s) + X_L(-s) \]

\[ x_R(t) = x(t)u(t) \quad \sigma_R = \lim_{t \to \infty} \frac{1}{t} \log |x(t)| \]
\[ x_L(t) = x(-t)u(-t) \quad \sigma_L = \lim_{t \to \infty} \frac{1}{t} \log |x(-t)| \]

Converges iff \( s > \sigma_R \) and \( -s > \sigma_L \).
A Theorem

Theorem. Assume that the following two limits exist:

\[ \sigma_R = \lim_{t \to \infty} \frac{1}{t} \log |x(t)| \]
\[ \sigma_L = \lim_{t \to -\infty} \frac{1}{t} \log |x(t)|. \]

If \( \sigma_R < \text{Re}(s) < \sigma_L \), then \( X(s) \) converges absolutely, and in fact \( X(s) \) is an analytic function in this region. On the other hand, if \( \text{Re}(s) < \sigma_R \) or if \( \text{Re}(s) > \sigma_L \), then \( X(s) \) diverges. This result is summarized by saying that \( \{ s : \sigma_R < \text{Re}(s) < \sigma_L \} \) is the region of convergence for the Laplace transform \( X(s) \).
Example

Example. Suppose that $f(t)$ is given by

$$x(t) = \begin{cases} 
  e^{-2t + \sqrt{t}} & \text{if } t \geq 0 \\
  te^{3t} & \text{if } t < 0.
\end{cases}$$

Then we have $\sigma_R = -2$ and $\sigma_L = 3$, so that the ROC for this signal is $-2 < |z| < 3$. 
Example. Let $f(t)$ be a causal signal defined by

$$f(t) = \begin{cases} 
  e^{3t} & \text{if } t \text{ is an even integer} \\
  e^{2t} & \text{if } t \text{ is not an integer} \\
  e^t & \text{if } t \text{ is an odd integer.}
\end{cases}$$

Then the limit in (‘eq:absoc’) doesn’t exist, since $\frac{1}{t} \log |f(t)|$ oscillates between 1 and 3. But the ROC is $\{s : \text{Re}(s) > 2\}$. 