Shannon's Version

• The Sampling Theorem: “If a function of time $f(t)$ is limited to the band $[0, W)$ Hz. it is completely specified by giving its ordinates at a series of discrete points spaced $\frac{1}{2W}$ seconds apart in the manner indicated by the [following formula]...

\[
f(t) = \sum_{n=-\infty}^{\infty} X_n \text{sinc}(2Wt - n)
\]

where

\[
X_n = f \left( \frac{n}{2W} \right).
\]

• In short: If the signal bandwidth is $W$ Hz., it can be communicated at a rate of $2W$ samples per second.
The Sinc and the Box

\[ \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \]

\[ \text{Box}(x) = \begin{cases} 
1 & \text{if } |x| \leq 1/2 \\
0 & \text{else} 
\end{cases} \]
Sinc-Box Duality (Third Way)

\[
sinc \left( \frac{t}{t_s} \right) \quad \mathcal{F} \quad t_s \text{ Box} \left( \frac{\omega}{\omega_s} \right) \quad (t_s \omega_s = 2\pi)
\]

\[
sinc \left( \frac{t - n}{t_s} \right) \quad \mathcal{F} \quad e^{-j\omega nt_s} t_s \text{ Box} \left( \frac{\omega}{\omega_s} \right)
\]
Sinc-Box Interpolation

Let $f(t, t_s)$ be the signal obtained by sampling $f(t)$ at $t_s$-second intervals, and then using “sinc interpolation” to reconstruct $f(t)$.

$$f(t, t_s) := \sum_{n=-\infty}^{\infty} f(nt_s) \text{sinc}(\frac{t}{t_s} - n)$$

$$F(j\omega, j\omega_s) := t_s \text{Box} \left( \frac{\omega}{\omega_s} \right) \sum_{n=-\infty}^{\infty} f(nt_s)e^{-j\omega nt_s}$$
Example
The Sampling Theorem

Sampling Theorem. *If* $f(t)$ *has bandwidth* $< \omega_s$, *then it can be reconstructed by its* $t_s$-*samples. Even stronger: If* $F(j\omega) = \text{Box} \left( \frac{\omega}{\omega_s} \right) F(j\omega)$, *then* $f(t) = f(t, t_s)$.
“You want proof? I'll give you proof!”
Proof of the Sampling Theorem

Define a periodic continuation of $F(j\omega)$:

$$\tilde{F}(j\omega) = \sum_{n=-\infty}^{\infty} F(j(\omega - n\omega_s)),$$

periodic of period $\omega_s$. Then

$$F(j\omega) = \text{Box}\left(\frac{\omega}{\omega_s}\right) \tilde{F}(j\omega)$$
Proof of the Sampling Theorem

Since $\tilde{F}(j\omega)$ is periodic of period $\omega_s$, by the First Way, $\tilde{F}(j\omega)$ can be expressed as a Fourier series

\begin{equation}
\tilde{F}(j\omega) = \sum_{n=-\infty}^{\infty} a[n] \exp(jnt_s\omega)
\end{equation}

with Fourier series coefficients

\[ a[n] = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} \tilde{F}(j\omega) \exp(-jnt_s\omega)d\omega \]

\[ = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} F(j\omega) \exp(-jnt_s\omega)d\omega \]

\[ = \frac{1}{\omega_s} \int_{-\infty}^{\infty} F(j\omega) \exp(-jnt_s\omega)d\omega \]

\[ = \frac{2\pi}{\omega_s} f(-nt_s) = t_s f(-nt_s) \]
Proof of the Sampling Theorem

Thus by (.1)

\[ \tilde{F}(j\omega) = t_s \sum_{n=-\infty}^{\infty} f(nt_s)e^{-jnt_s\omega} \]

\[ F(j\omega) = \sum_{n=-\infty}^{\infty} f(nt_s)t_s \text{Box} \left( \frac{w}{w_s} \right) e^{-jnt_s\omega} \]

\[ \uparrow \mathcal{F} \]

\[ f(t) = \sum_{n=-\infty}^{\infty} f(nt_s) \text{sinc}(t/t_s - n) \]

\[ = f(t, t_s) \]
“Impulse Train”

Sampling—Chapter 7's Method

\[ f_p(t) = \sum_{n=-\infty}^{\infty} (2\pi f(nt_s))\delta(t - nt_s) \]

\[ F_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \]

\[ = \sum_{n=-\infty}^{\infty} f(nt_s)e^{-j\omega nt_s} \]

\[ = \tilde{F}(j\omega) \quad \text{etc. as before} \]
The Skeleton in Nyquist's Closet

However, no signal of practical interest can be truly bandlimited!
The Skeleton in Nyquist's Closet

However, no signal of practical interest can be truly bandlimited!
“You want proof? I'll give you proof!”
Proof

If \( f(t) \) is bandlimited, its Fourier transform \( F(\omega) \) is zero outside \([-\omega_s, \omega_s]\), and so we have

\[
(1) \quad f(t) = \frac{1}{2\pi} \int_{-\omega_s}^{+\omega_s} F(\omega) e^{j\omega t} d\omega
\]

for all real numbers \( t \). If we extend the definition of \( f(t) \) to all complex numbers by defining, for each complex \( z \),

\[
(2) \quad f(z) = \frac{1}{2\pi} \int_{-\omega_s}^{+\omega_s} F(\omega) e^{j\omega z} d\omega,
\]

then it is possible to show that this integral exists for all \( z \), and that in fact \( f(z) \) is an analytic function of \( z \), for all \( z \).
Proof

The set of zeros of $f(z)$ can have no limit point, unless $f(z)$ is identically zero. It follows that the original signal $f(t)$ cannot be time limited, since if say $f(t) = 0$ for all $t < 0$, it would follow that $f(z)$, and so also $f(t)$ is identically zero. And since any “real” signal must be time-limited, it follow that no “real” signal (except the identically zero signal) can be strictly bandlimited.
And Now for ... the DFT
Computing the Continuous-Time Fourier Transform Numerically

- Select time and frequency resolution:

\[ \Delta t = \frac{T}{N}, \quad \Delta \omega = \frac{2\pi}{T} \]

\[ F(j\omega) = \int_0^T f(t)e^{-j\omega t} dt \approx \Delta t \sum_{n=0}^{N-1} f(n\Delta t)e^{-j\omega n\Delta t} \]

\[ F(jk\Delta \omega) \approx \Delta t \sum_{n=0}^{N-1} f(n\Delta t)(w_N)^{nk} \]
• If $x[n]$ has period $N$, and $w_N = \exp(-2\pi j/N)$, then $x[n] \leftrightarrow \hat{x}[k]$:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}[k](w_N)^{-kn}$$

$$\mathcal{F} \uparrow$$

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n](w_N)^{nk}.$$
$W_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

\[ \hat{x}[0] = x[0] + x[1] \]
\[ \hat{x}[1] = x[0] - x[1] \]
N=4 DFT Requires 16 COPS

\[ W_4 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & j & -1 & -j \\
1 & -1 & 1 & -1 \\
1 & -j & -1 & j
\end{pmatrix} \]

And Now for … the FFT
$N=4$ DFT Requires Only 9 COPS:
Hence the Adjective “Fast”
The FFT Trick
(Decimation in Time Style)

\[ \hat{x}[k] = \sum_{n=0}^{N-1} x[n](w_N)^{nk} \]

\[ = \sum_{m=0}^{N/2-1} x[2m](w_N)^{2mk} + \sum_{m=0}^{N/2-1} x[2m+1](w_N)^{(2m+1)k} \]

\[ = \sum_{m=0}^{N/2-1} x[2m](w_N^2)^{mk} + (w_N)^{k} \sum_{m=0}^{N/2-1} x[2m+1](w_N^2)^{mk} \]
The FFT Trick, Continued

\[ \hat{x}[k] = \sum_{m=0}^{N/2-1} x[2m] (w_{N/2})^{mk} \mod N/2 \]

\[ + (w_N)^k \sum_{m=0}^{N/2-1} x[2m + 1] (w_{N/2})^{mk} \mod N/2 \]

\[ = \hat{x}_0[k \mod N/2] + (w_N)^k \hat{x}_1[k \mod N/2] \quad \text{for } k = 0, \ldots, N - 1 \]
A Recursive FFT Circuit

Figure 1. Building a $F_N$ from two $F_{N/2}$’s. This shows that $\text{COPS}[N] \leq 2 \text{COPS}[N/2] + 3N/2$
FFT Complexity

\( \text{COPS}[N] = 2 \text{COPS}[N/2] + 3N/2 \)

\( \text{COPS}[2] = 2 = 1.0N \)
\( \text{COPS}[4] = 10 = 2.5N \)
\( \text{COPS}[8] = 32 = 4N \)
\( \text{COPS}[16] = 88 = 5.5N \)

\( \text{COPS}[N] = N(1.5 \log_2 N - 0.5) \)
The FFT Trick (Decimation in Frequency Style, Even Subscripts)

\[
\hat{x}[2k] = \sum_{n=0}^{N-1} x[n](w_N)^{2nk}
\]

\[
= \sum_{n=0}^{N-1} x[n](w_{N/2})^{nk}
\]

\[
= \sum_{n=0}^{N/2-1} x[n](w_{N/2})^{nk} + \sum_{n=N/2}^{N-1} x[n](w_{N/2})^{nk}
\]

\[
= \sum_{n=0}^{N/2-1} x[n](w_{N/2})^{nk} + \sum_{n=0}^{N/2-1} x[n + N/2](w_{N/2})^{nk}
\]

\[
= \sum_{n=0}^{N/2-1} (x[n] + x[n + N/2])(w_{N/2})^{nk}
\]
The FFT Trick, Decimation in Frequency Style, Odd Subscripts

\[
\hat{x}[2k + 1] = \sum_{n=0}^{N-1} x[n](w_N)^{n(2k+1)}
\]

\[
= \sum_{n=0}^{N-1} x[n](w_N)^n(w_{N/2})^{nk}
\]

\[
= \sum_{n=0}^{N/2-1} x[n](w_N)^n(w_{N/2})^{nk} + \sum_{n=N/2}^{N-1} x[n](w_N)^n(w_{N/2})^{nk}
\]

\[
= \sum_{n=0}^{N/2-1} x[n](w_N)^n(w_{N/2})^{nk} + \sum_{n=0}^{N/2-1} x[n + N/2](w_N)^{n+N/2}(w_{N/2})^{nk}
\]

\[
= \sum_{n=0}^{N/2-1} (w_N)^n(x[n] - x[n + N/2])(w_{N/2})^{nk}
\]
Decimation in Frequency

Figure 2. Compare to Figure 1