Solutions to Final Examination

Problem 1. Give an example of a stable discrete-time system whose system function has a pole at $z = 2$.

Solution: The system must be anti-causal. $h[n] = -2^n u[-n - 1]$, $H(z) = \frac{1}{1 - 2z^{-1}}$.

Problem 2. A certain causal discrete-time system has system function $H(z) = 1/(1 - 2z^{-1} + z^{-2})$. What is the $n$th term of the impulse response $h[n]$ of the system?

Solution: $H(z) = \frac{1}{(1 - z^{-1})^2}$, $h[n] = (n + 1)u[n]$

Problem 3. Let $f(t) = \text{sinc}^2(t)$. What is the Fourier Transform $F(j\omega)$?

If $B(j\omega)$ is the FT of $\text{sinc}(t)$, then $\frac{1}{2\pi}B(j\omega) * B(j\omega)$ is the FT of $f(t)$. But from the notes, $B(j\omega) = \text{Box}(\omega/2\pi)$ Thus

$$F(j\omega) = \begin{cases} \frac{2\pi - |\omega|}{2\pi} & \text{if } |\omega| < 2\pi \\ 0 & \text{else} \end{cases}$$

Problem 4. Consider the following difference equation.

$$y[n] = x[n] + 2y[n - 1].$$

Assume $x[n] = y[n] = 0$ for $n < 0$. Find the shortest possible nonzero input signal $x[n]$ that produces a finite output signal. (Note: A signal $x[n]$ is said to be finite if $x[n] = 0$ for all $n \geq n_0$.)

Solution: $Y(z) = \frac{X(z)}{1 - 2z^{-1}}$. Hence a shortest input signal is $X(z) = 1 - 2z^{-1}$, i.e., $(1, -2, 0, 0, \ldots)$. The most general such is $X(z) = z^m - 2z^{m-n}$.

Problem 5. Find a simplified expression for the signal

$$f(t) = \sum_{n=-\infty}^{+\infty} \text{sinc}(t - n).$$

Solution:

$$f(t) = 1 \quad \text{for all } t$$

Problem 6. How many complex additions and multiplications, at most, are required to perform a “fast” DFT for $N = 6$?

Solution: Using the formula $\text{COP}(N) = 2\text{COP}(N/2) + 3N/2$, and $\text{COP}(3) = 10$, a conservative estimate is $\text{COP}(6) \leq 29$. 

Problem 7. Find a continuous-time signal whose Laplace transform’s ROC is \( \{ s : 1 < \text{Re} \ s < 2 \} \).

Solution: \( x(t) = e^t u(t) - e^{2t} u(-t) \).

Problem 8. Let \( u[n] \) be the unit step function. Find a causal signal \( g[n] \) that satisfies the equation

\[
g[n] * g[n] = u[n]
\]

Note: Ideally your answer will be of the form “\( g[n] = \) (explicit formula).”

Solution: Here \( H(z) = \sum_{n \geq 0} z^{-1} = 1/(1 - z^{-1}) \), so that \( G(z) = (1 - z^{-1})^{-1/2} \). But as we discussed in class, we can apply the binomial theorem, even when the exponent is negative, so that

\[
g(z) = (1 - z^{-1})^{-1/2} = \sum_{n \geq 0} (-1)^n \binom{-1/2}{n} z^{-n}.
\]

Thus for \( n \geq 0 \),

\[
g[n] = (-1)^n \binom{-1/2}{n} = \frac{1}{2^n} \binom{2n}{n} = \prod_{k=1}^{n} \left( 1 - \frac{1}{2k} \right).
\]

For example \( g[1] = 1/2, g[2] = 3/8, g[3] = 5/16, \) etc.

Alternative formulas that were accepted include

\[
g[n] = \frac{1}{2} \left( 1 - \sum_{k=1}^{n-1} g[k] g[n-k] \right).
\]

\[
g[n] = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{j\omega n}}{(1 - e^{-j\omega})^{1/2}} d\omega.
\]

Note: Many students attempted to apply the formula at the bottom of Table 5.2, with \( r = 1/2 \). This formula only works for positive integer values of \( r \). In particular, with \( r = 1/2 \), the quantity \((-1/2)!\) is not defined. If you used this approach, you probably lost 5 points.