THEORY OF CYCLIC FILTER BANKS

P. P. Vaidyanathan and Ahmet Kirac

Dept. Electrical Engr., Caltech, Pasadena, CA 91125

Abstract. [1] We introduce the fundamentals of cyclic multirate systems and filter banks and present a number of important differences between the cyclic and noncyclic (traditional) cases. Some of the additional freedom offered by cyclic systems is pointed out, and a number of open issues are summarized.

1. INTRODUCTION

Digital filter banks [1]–[2] for finite length signals have been considered by a number of authors [3]-[8]. Smith and Eddins [3] introduced this idea for image coding, and among other significant results, proposed replacing linear filtering operations with cyclic (or circular) convolution. They also noted that many results from the noncyclic case carry over routinely. In this paper we draw inspiration from these references to develop the theory for cyclic filter banks, and highlight the significant differences from noncyclic case. A cyclic(L) filter bank (with the letter L often omitted) has an input $x(n)$ defined only for $0 \leq n \leq L-1$ (this could be a symmetrically extended version [3] of a shorter finite length signal; but this is not the main point here).

Fig. 1 shows the M-channel cyclic(L) filter bank. We assume throughout that

$$L = KM, \quad K = \text{ integer.} \qquad (1)$$

The filters have impulse responses $h_i(n)$ and $f_i(n)$ confined to $0 \leq n \leq L-1$, and their L-point DFTs are denoted as $H_i(k)$ and $F_i(k)$. All convolutions are cyclic(L). So we can regard $x(n)$ to be periodic (or cyclic) with period L. Since a cyclic(L) signal can be represented by L samples of the Fourier transform (i.e., the DFT coefficients) rather than the entire Fourier transform, it implies more freedom in the design. The paraunitary and power complementary properties, and even the linear phase property are more relaxed, as they are imposed only on a discrete frequency grid.

2. BASICS OF CYCLIC MULTIRATE SYSTEMS

The notation $W_L \triangleq e^{-j2\pi/L}$ (or just W) will be used throughout. Since the frequency variable ω is replaced by the discrete version $2\pi k/L$, the quantity $W_L^k = e^{-j2\pi k/L}$ is the **unit-delay operator** analogous to $z^{-1} = e^{-j\omega}$. The **cyclic decimator**, denoted by $\downarrow M$ in Fig. 1, has the input-output relation $y(n) = x(Mn)$. With $x(n)$ regarded as cyclic(L), the output $y(n)$ is cyclic(K). Thus all the subband signals $v_i(n)$ are cyclic(K). For the decimator, let $X(k)$ denote the L-point DFT of $x(n)$ and $Y(k)$ the K-point DFT of its output $y(n)$, i.e.,

$$X(k) = \sum_{n=0}^{L-1} x(n)W_L^{nk}, \quad 0 \leq k \leq L-1 \qquad (2)$$

and $Y(k) = \sum_{n=0}^{K-1} y(n)W_K^{nk}$, $0 \leq k \leq K-1$. It can be verified that $Y(k) = \sum_{i=0}^{M-1} X(Ki+k)/M$, for $0 \leq k \leq K-1$. The **cyclic expander**, denoted by $\uparrow M$ in Fig. 1, has a periodic(K) input $x(n)$ and periodic(L) output $y(n)$ related by

$$y(n) = \begin{cases} x(n/M), & n = \text{mul. of } M \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

The corresponding DFT relation is $Y(k) = X(k)$ for $0 \leq k \leq L-1$. Since $X(k)$ are the K-point DFT coefficients of $x(n)$, the L-point DFT $Y(k)$ has the shorter period $K = L/M$.

Polyphase representation. The L-point DFT of a cyclic(L) impulse response $h(n)$ can be expressed as

$$H(k) = \sum_{n=0}^{L-1} h(n)W_L^{nk} = \sum_{\ell=0}^{M-1} (W_L^k)^{\ell} E_{\ell}(k) \qquad (4)$$

where $E_\ell(k) = \sum_{n=0}^{K-1} h(Mn+\ell)W_K^{kn}$, for $0 \leq k \leq L-1$. Thus, $E_\ell(k)$ is the K-point DFT of the ℓth *polyphase component* $e_\ell(n) \triangleq h(Mn+\ell)$. From the definition of $E_\ell(k)$ we see that it is cyclic(K). Eq. (4) is analogous to the traditional Type 1 polyphase decomposition $H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$. Fig. 2(b) shows a decimation filter redrawn in polyphase form. Similarly the Type 2 polyphase form is given by $H(k) = \sum_{\ell=0}^{M-1} W_L^{-k\ell} R_\ell(k)$. Since the polyphase components $E_\ell(k)$ have a smaller period K, they can be relocated to the right of the decimators as in Fig. 2(c) (similar to the use of noble identities in the noncyclic case).

[1]Work supported in parts by the Office of Naval Research Grant N00014-93-1-0231, Tektronix, Inc., and Rockwell Intl.

A caution: We have used the same notation $E_\ell(k)$ in Figs. 2(b) and 2(c). This is regarded as a K-point DFT in Fig. 2(c) and an L-point DFT (with values repeating after a shorter period K) in Fig. 2(b).

Return now to the analysis/synthesis system of Fig. 1. With the filters $H_i(k)$ represented in Type 1 polyphase form and the filters $F_i(k)$ in Type 2 form, we have the equivalent representation of Fig. 3 where $\mathbf{E}(k)$ and $\mathbf{R}(k)$ are the **polyphase matrices** of the cyclic filter bank. These should be interpreted as K-point DFTs, e.g., $\mathbf{E}(k) = \sum_{n=0}^{K-1} \mathbf{e}(n) W_K^{kn}$. The filter bank has the PR property iff $\mathbf{R}(k)\mathbf{E}(k) = \mathbf{I}$ for all k.

Orthonormality. We define the M-band cyclic filter bank to be orthonormal if the matrix $\mathbf{E}(k)$ is unitary for all k. This property is referred to as the **cyclic-paraunitary** property. The perfect reconstruction property then reduces to $\mathbf{R}(k) = \mathbf{E}^\dagger(k)$, or, in terms of impulse responses,

$$f_i(n) = h_i^*(-n) \tag{5}$$

(with arguments interpreted modulo L). The DFTs are correspondingly related as $F_i(k) = H_i^*(k)$. Assuming $\widehat{x}(n) = x(n)$ in Fig. 1 we have

$$x(n) = \sum_{i=0}^{M-1} \sum_{\ell=0}^{K-1} v_i(\ell) f_i(n - \ell M), \quad 0 \le n \le L-1.$$

The basis functions are $\eta_{i,\ell}(n) \stackrel{\Delta}{=} f_i(n - \ell M), \quad 0 \le n \le L-1$, where $0 \le i \le M-1, 0 \le \ell \le K-1$. Thus there are $MK = L$ basis functions $\eta_{i,\ell}(n)$. It can be shown that the unitarity of $\mathbf{R}(k)$ is equivalent to orthonormality of the basis $\eta_{i,\ell}(n)$. This orthonormality can be reexpressed as $\sum_{n=0}^{L-1} f_i(n) f_m^*(n - M\ell) = \delta(i-m)\delta(\ell)$. As in traditional filter banks, orthonormality of the cyclic filter bank implies the unit energy property $\sum_{n=0}^{L-1} |f_i(n)|^2 = 1$ and the power complementary property $\sum_{i=0}^{M-1} |F_i(k)|^2 = M$.

3. CYCLIC VERSUS NONCYCLIC FILTER BANKS

In the cyclic(L) case, any transfer function can be expressed in the form $H(k) = \sum_{n=0}^{L-1} h(n) W_L^{kn}$. The noncyclic counterpart of $H(k)$ is defined as $H_{non}(z) = \sum_{n=0}^{L-1} h(n) z^{-n}$. It is an **interpolated version** in the frequency domain, since $H(k)$ = samples of $H_{non}(z)$ at $z = e^{j2\pi k/L}$. Similarly the noncyclic counterpart of $\mathbf{E}(k) = \sum_{n=0}^{K-1} \mathbf{e}(n) W_K^{kn}$ (Fig. 3) is $\mathbf{E}_{non}(z) = \sum_{n=0}^{K-1} \mathbf{e}(n) z^{-n}$. If $\mathbf{E}_{non}(z)$ is paraunitary (PU) it readily follows that $\mathbf{E}(k)$ is cyclic-PU because each k corresponds to a special z on the unit circle. But the converse does not hold as we shall see. Thus the cyclic-PU property is less stringent than traditional PU. More generally, if we impose a certain constraint on the cyclic system, it does not mean that the noncyclic counterpart has to satisfy the same constraint. To demonstrate, consider the second order cyclic(4) transfer function

$$G(k) = 0.5 + 0.5(j-1)W_4^k + 0.5jW_4^{2k}$$

Using the facts that $W_4^4 = 1$ and $W_4^2 = -1$, it is readily verified that $|G(k)|^2 = 1$ for all k. Thus $G(k)$ is **allpass in the cyclic(4) sense.** However, the noncyclic version $G_{non}(z) = 0.5 + 0.5(j-1)z^{-1} + 0.5jz^{-2}$ which is an FIR filter, is evidently not allpass. If we now construct the $M \times M$ polyphase matrix $\mathbf{E}(k) = G(k)\mathbf{I}_M$ (for arbitrary M) then it is cyclic-PU, but the noncyclic version $\mathbf{E}_{non}(z) = \mathbf{G}_{non}(z)\mathbf{I}$ is not PU. As a second example, consider the cyclic(3) analysis bank

$$\mathbf{H}(k) = \begin{bmatrix} H_0(k) \\ H_1(k) \\ H_2(k) \end{bmatrix} = \mathbf{a}_0 + \mathbf{a}_1 W_3^k + \mathbf{a}_2 W_3^{2k}$$

where $\mathbf{a}_i$ are the column-vectors given by

$$\mathbf{a}_0 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{a}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

It can be verified that $\mathbf{H}^\dagger(k)\mathbf{H}(k) = 1$, that is, the three transfer functions $H_0(k), H_1(k)$ and $H_2(k)$ satisfy the **cyclic power complementary** property $|H_0(k)|^2 + |H_1(k)|^2 + |H_2(k)|^2 = 1$. But the noncyclic version $\mathbf{H}_{non}(z) = \sum_{n=0}^{2} \mathbf{a}_n z^{-n}$ is not power complementary, that is, $\widetilde{\mathbf{H}}_{non}(z)\mathbf{H}_{non}(z) \neq 1$.

Cyclic allpass filters. A cyclic(L) allpass filter satisfies $H(k) = e^{j\phi(k)}$ for $0 \le k \le L-1$. We can always write $H(k) = c(\sum_{n=0}^{N} b_{N-n}^* W_L^{kn})/(\sum_{n=0}^{N} b_n W_L^{kn})$ with $|c| = 1$. For example, we can let $N = L-1$ and set $e^{-j\phi(k)/2} = \sum_{n=0}^{L-1} b_n W_L^{kn}$. The coefficients b_n are essentially the inverse DFT coefficients of $e^{-j\phi(k)/2}$ and can readily be identified. *More interesting is the open problem of obtaining the rational form with smallest order* N. Now consider the $L \times L$ **circulant matrix H** whose first row is $h(0), h(1), h(2) \ldots$ We know that circulant matrices are diagonalized by the DFT matrix (which is unitary) and that the eigenvalues are the DFT coefficients $H(k)$. Using this we see that the *allpass property is equivalent to unitariness of* $\mathbf{H}$.

Factorization of cyclic paraunitary systems. Noncyclic PU systems can be factored [2] in terms of the building blocks $\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + z^{-1}\mathbf{u}_i\mathbf{u}_i^\dagger$, where $\mathbf{u}_i^\dagger\mathbf{u}_i = 1$. But in the cyclic case, factorization in terms of $(\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + W_L^k\mathbf{u}_i\mathbf{u}_i^\dagger)$ is not always possible. When it is possible, we can replace W_L^k with z^{-1} in the factorization, and obtain a noncyclic PU counterpart $\mathbf{E}_{non}(z)$.

Determinant of cyclic paraunitary system. If $\mathbf{E}(k)$ is cyclic PU, then $[\det \mathbf{E}(k)] = e^{j\phi(k)}$ which is allpass. In the noncyclic FIR case the degree of $[\det \mathbf{E}_{non}(z)]$

is equal to the degree of $\mathbf{E}_{non}(z)$, but the same is not true in the cyclic case. Thus let

$$\mathbf{E}(k) = \begin{bmatrix} \cos\theta(k) & \sin\theta(k) \\ -\sin\theta(k) & \cos\theta(k) \end{bmatrix}, \quad 0 \le k \le L-1 \quad (6)$$

which is cyclic PU with [det $\mathbf{E}(k)$] = 1. The determinantal degree is zero, regardless of the degree of $\mathbf{E}(k)$ (which is the minimum number of unit delay elements W^k required in the implementation).

Nyquist and Linear-Phase Properties. Consider a cyclic(6) transfer function

$$H_0(k) = \left(1 + W_6^k - W_6^{2k} + W_6^{3k} + W_6^{4k}\right)/\sqrt{5}.$$

We can regard this as FIR in the sense that $h(n)$ is nonzero only on a subset of points in $0 \le n \le L-1$. (In the cyclic case this is the only FIR-definition that makes sense). The symmetry of the impulse response implies the linear-phase property, that is, $H_0(k) = W_6^{2k} \times R(k)$ where $R(k)$ is real. Now consider $G(k) = |H_0(k)|^2$. Its cyclic(6) impulse response $g(n)$ satisfies the **halfband** property $g(2n) = \delta(n)$. Equivalently $H(k)$ is **power symmetric** [**2**] in the cyclic sense:

$$|H_0(k)|^2 + |H_0(3+k)|^2 = 2, \quad 0 \le k \le 5. \quad (7)$$

So we have a cyclic(6) filter $H_0(k)$ which is both linear-phase and power-symmetric. This is not possible in the noncyclic FIR case [**2**].

Using the above example we can construct a two-channel cyclic(6) orthonormal filter bank where the filters are nontrivial linear phase filters. (Such constructions are not possible in the noncyclic FIR case [**2**].) For this choose $h_0(n)$ as above, and the remaining three filters as $h_1(n) = (-1)^n h_0^*(1-n)$, $f_i(n) = h_i^*(-n)$. That is, $H_1(k) = -W_L^k H_0^*(k-K)$ where $K = L/2$, and $F_i(k) = H_i^*(k)$ (recall all arguments are interpreted modulo-6). This an example of a **cyclic(6) version of the CQF design**.

More on the CQF Design. Given a cyclic(L) halfband filter $g(n)$ with the property $G(k) \ge 0$, we can construct infinitely many cyclic(L) two channel orthonormal filter banks by choosing $H_0(k)$ as a spectral factor

$$H_0(k) = e^{j\phi(k)}\sqrt{G(k)}, \quad (8)$$

and the remaining three filters according to the CQF equations above. Recall that for noncyclic FIR filters, the usual definition of spectral factors allows only finitely many phase responses, and the spectral factors all have the same length $N+1$. In Eq. (8) however, the phase response $\phi(k)$ of the spectral factor $H_0(k)$ **is arbitrary.** In particular, the choice $\phi(k) = 0$ would yield linear phase analysis filters $H_0(k)$ and $H_1(k)$ with good frequency responses (if $G(k)$ is a good lowpass filter). However, even if $G(k)$ is cyclic(L) FIR (in the sense that $g(n) = 0$ for $N < n < L-N$) the cyclic spectral factor $H_0(k)$ may not be FIR (i.e., $h_0(k)$ could be nonzero for all n). See Fig. 4. *An open question here is, what is the most general form* of the phase response $\phi(k)$ which ensures that $H_0(k)$ is FIR (possibly with length $N+1$).

4. CYCLIC DIFFERENCE EQUATIONS

Consider a cyclic(L) transfer function of the form $H(k) = (a_0 + a_1 W_L^k)/(1 - bW_L^k)$. The input and output of this system are constrained by $(1 - bW_L^k)Y(k) = (a_0 + a_1 W_L^k)X(k)$. By taking inverse DFT we obtain

$$y(n) = by(n-1) + a_0 x(n) + a_1 x(n-1) \quad (9)$$

Since the time-indices are interpreted modulo L, this is a cyclic difference equation. To demonstrate what is involved, let $L = 3$. Repeated use of the d.e. (and the facts that $y(2) = y(-1)$, $y(3) = y(0)$ etc.) therefore yields the conclusion that $(1-b^3)y(0) = \alpha_0 x(0) + \alpha_1 x(1) + \alpha_2 x(2)$ for some constants α_i. Thus the initial condition $y(0)$ is not arbitrary, but is uniquely determined as long as $b^3 \ne 1$, that is, as long as $b \ne W_3^i$ for any i. This is equivalent to the obvious requirement that the denominator in $H(k)$ does not become zero for any k. We can then write $Y(k) = H(k)X(k)$, and the inverse DFT $y(n)$ is determined for all n. More generally, consider the cyclic(L) transfer function $H(k) = (\sum_{n=0}^{N} a_n W_L^{kn})/(1 + \sum_{n=1}^{N} b_n W_L^{kn})$. With the implicit assumption that the denominator does not vanish for any k, the output is fully determined by the input. In particular the *"initial" condition is predetermined, and is not arbitrary.*

To compute $y(n)$ efficiently, we can use a recursive d.e. similar to (9) provided we know how to identify the (unique) initial condition. A convenient way is the state space apprach. From the direct-form structure of Fig. 5 we can identify a set of N state variables $v_i(n)$ (outputs of the unit delay elements W_L^k) and obtain equations of the form

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}x(n) \quad (10)$$

and $y(n) = \mathbf{C}\mathbf{v}(n) + Dx(n)$ where $\mathbf{v}(n)$ is the state vector with components $v_i(n)$. Repeated use of (10) yields $\mathbf{v}(L) = \mathbf{A}^L\mathbf{v}(0)$ + *a linear combination of samples of* $x(n)$. Since all the time-indices are interpreted modulo-L, we have $\mathbf{v}(L) = \mathbf{v}(0)$, so $(\mathbf{I} - \mathbf{A}^L)\mathbf{v}(0)$ = linear combination of samples of $x(n)$. Thus we can identify the initial state $\mathbf{v}(0)$, if $\mathbf{I} - \mathbf{A}^L$ is nonsingular (i.e., no eigenvalue of $\mathbf{A}$ has the form W_L^i for any i).

As in the noncyclic case the transfer function $H(k)$ can be expressed as $H(k) = D + \mathbf{C}(W_L^{-k}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, but the impulse response is $h(n) = \mathbf{C}\mathbf{A}^{n-1}(\mathbf{I} - \mathbf{A}^L)^{-1}\mathbf{B}$ for $0 < n < L$ and $h(0) = D + \mathbf{C}\mathbf{A}^{L-1}\left(\mathbf{I} - \mathbf{A}^L\right)^{-1}\mathbf{B}$. This is because the initial condition $\mathbf{v}(0)$ is predetermined and cannot be set to zero.

The **system matrix** for an LTI cyclic implementation is defined as $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$. If this is unitary, one can verify (as in the noncyclic case [2]) that $\mathbf{E}(k)$ is cyclic PU. However the converse is not true in the cyclic case: even if $\mathbf{E}(k)$ is PU, there may not exist a minimal structure with unitary system matrix. Whenever it does, we can find an interpolated PU matrix $\mathbf{E}_{non}(z)$ by replacing W_L^k by z^{-1} in the expression $H(k) = D + \mathbf{C}(W_L^{-k}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, and then factorize it. In the factored form if we replace z^{-1} with W^k, we obtain a factorization of $\mathbf{E}(k)$.

5. CONCLUDING REMARKS

Perhaps the question of greatest interest for future work is, how to exploit the extra freedom offered by the cyclic system, in the design of subband coders. Can this be exploited to obtain increased coding gain (or compression), or to obtain reduced complexity of implementation? These require further study. The notion of cyclic LTI systems also opens up other problems in the general arena of signal and system theory.

References

[1] Malvar, H. S. *Signal processing with lapped transforms,* Artech House, Norwood, MA, 1992.

[2] Vaidyanathan, P. P. *Multirate systems and filter banks,* Prentice Hall, 1993.

[3] Smith, M. J. T., and Eddins, S. L. "Analysis/synthesis techniques for subband image coding," *IEEE Trans. ASSP,* pp. 1446-1456, Aug. 1990.

[4] Karlsson, G. and Vetterli, M. "Extension of finite length signals for subband coding," Signal Processing, vol. 17, pp. 161-168, June 1989.

[5] Nuri, V., and Bamberger, R. H. "Size-limited filter banks for subband image compression", IEEE Trans. Image Proc., pp. 1317-1323, Sept. 1995.

[6] de Queiroz, R. L., and Rao, K. R. "On reconstruction methods for processing finite-length signals with paraunitary filter banks", IEEE Trans. Signal Proc., pp. 2407-2410, Oct. 1995.

[7] Chen, L., Nguyen, T. Q., and Chan K-W. "Symmetric extension methods for M-channel linear-phase perfect-reconstruction filter banks," IEEE Trans. Signal Proc., pp. 2505-2511, Nov. 1995.

[8] Kiya, H., Nishikawa, K., Sagawa, M. "Property of circular convolution for subband image coding," IEICE Trans. Fundamentals, pp. 852-860, July 1992.

[9] Smith, M. J. T., and Barnwell III, T. P. "Exact reconstruction techniques for tree-structured subband coders," *IEEE Trans.ASSP,* pp. 434-441, June 1986.

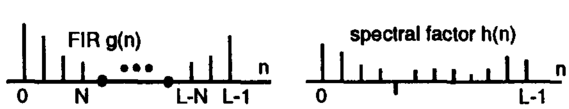

Fig. 4. A cyclic(L) FIR filter with non-FIR spectral factor.

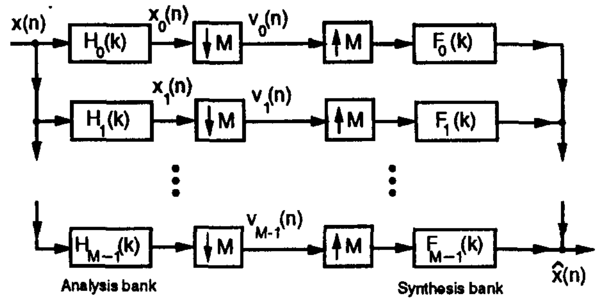

Fig. 1. The cyclic filter bank.

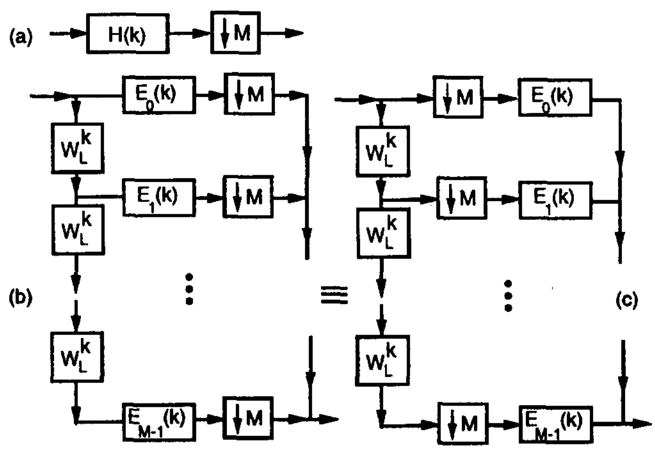

Fig. 2. (a) A decimation filter and (b),(c) polyphase forms.

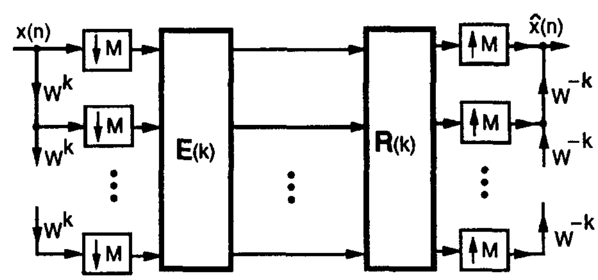

Fig. 3. Polyphase form of the cyclic filter bank.

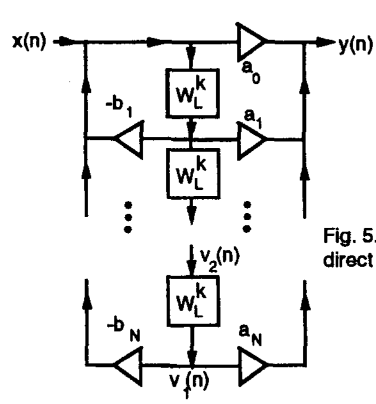

Fig. 5. The cyclic direct form structure.