High Order Super Nested Arrays

Chun-Lin Liu[1] and P. P. Vaidyanathan[2]

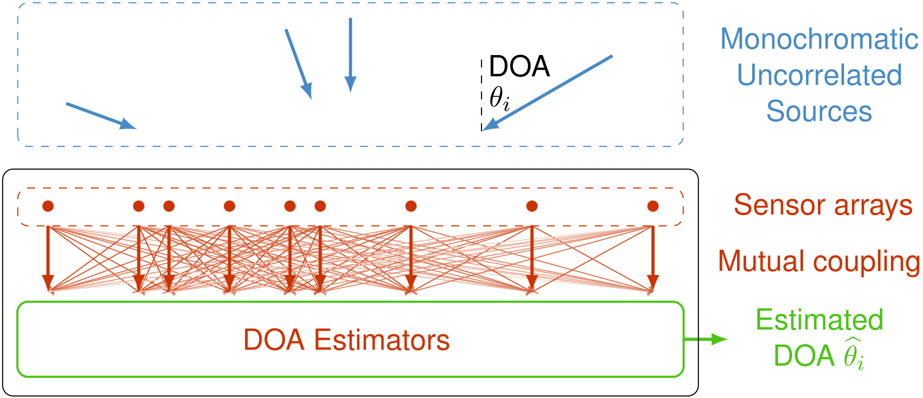
Dept. of Electrical Engineering, MC 136-93
California Institute of Technology,
Pasadena, CA 91125, USA
cl.liu@caltech.edu[1], ppvnath@systems.caltech.edu[2]

SAM 2016

Outline

Outline

DOA estimation in the presence of mutual coupling[1]



We will develop new sparse arrays with less mutual coupling.

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

ULA and sparse arrays

ULA (not sparse)

- Identify at most $N-1$ uncorrelated sources, given N sensors.[1]
- Can only find fewer sources than sensors.

Sparse arrays

1. Minimum redundancy arrays[2]
2. Nested arrays[3]
3. Coprime arrays[4]
4. Super nested arrays[5]

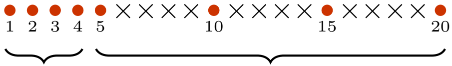
- Identify $O(N^2)$ uncorrelated sources with $O(N)$ physical sensors.
- More sources than sensors!

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.
[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968.
[3] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010.
[4] Vaidyanathan and Pal, *IEEE Trans. Signal Proc.*, 2011.
[5] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Nested arrays[1]

The nested array
$N_1 = 4,$
$N_2 = 4.$

$|\mathbb{S}| = N_1 + N_2 = 8$



Dense ULA
N_1 sensors
spacing 1

Sparse ULA
N_2 sensors
spacing $N_1 + 1$

Difference coarray

$\mathbb{D} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{S}\}$

$|\mathbb{D}| = O(N_1 N_2)$

For sufficient number of snapshots,
$(|\mathbb{U}| - 1)/2 = O(N_1 N_2)$ uncorrelated sources can be identified.
($\mathbb{U}$ = Central ULA part of $\mathbb{D}$)

[1] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010.

Outline

Super nested arrays[1]

1. Super nested arrays have the same number of sensors as nested arrays.
2. Super nested arrays have the same difference coarrays as nested arrays. In particular, no holes.
3. Super nested arrays are more sparse than nested arrays, i.e., super nested arrays have less mutual coupling.

Nested array $N_1 = 13, N_2 = 5.$

Super nested array $N_1 = 13, N_2 = 5.$

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

How to characterize mutual coupling in arrays?[1]

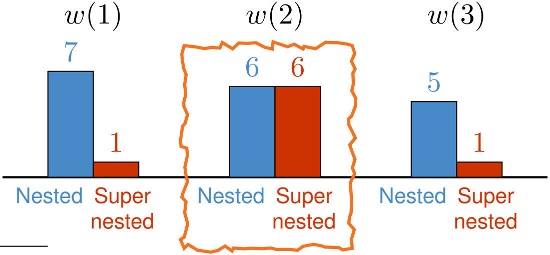
The weight function $w(m)$

The number of sensor pairs with separation m.

Nested array, $N_1 = N_2 = 7$

Super nested array, $N_1 = N_2 = 7$

More sparse
Less mutual coupling



[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Outline

Goal: Desired properties of super nested arrays

They should have the same number of sensors as nested arrays,

$$|\mathbb{S}_{\text{High order super nested}}| = |\mathbb{S}_{\text{Super nested}}| = |\mathbb{S}_{\text{Nested}}|.$$

They should have the same difference coarray as nested arrays,

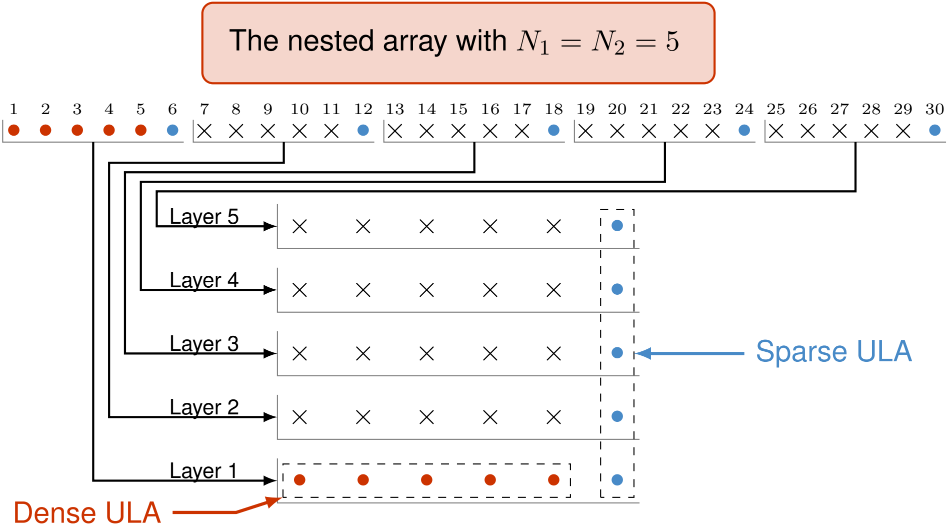
$$\mathbb{D}_{\text{High order super nested}} = \mathbb{D}_{\text{Super nested}} = \mathbb{D}_{\text{Nested}}.$$

(In particular, no holes)

They should be more sparse than nested arrays,

$$w_{\text{High order super nested}}(1) \leq w_{\text{Super nested}}(1) \leq w_{\text{Nested}}(1),$$
$$w_{\text{High order super nested}}(2) \leq w_{\text{Super nested}}(2) \leq w_{\text{Nested}}(2),$$

2D representations for 1D nested arrays[1]

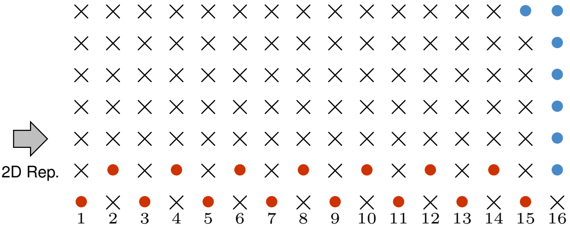


[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

High order super nested arrays

Second-order super nested array

$N_1 = 15,$
$N_2 = 7,$
$Q = 2$

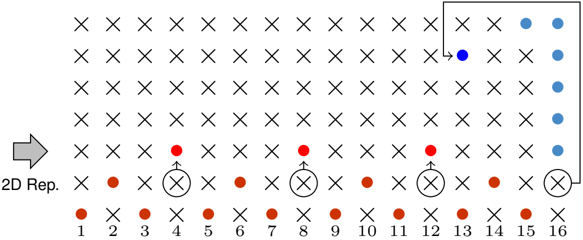


1D Rep.

1D Rep.

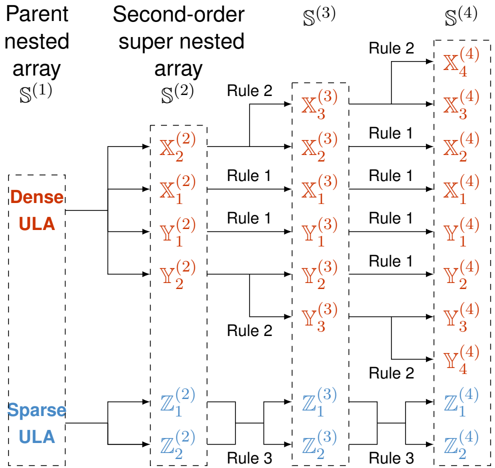
High-order super nested array

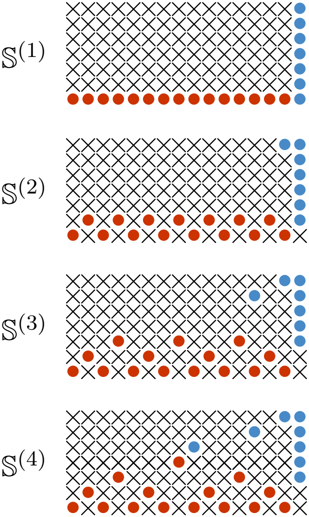
$N_1 = 15,$
$N_2 = 7,$
$Q = 3.$



The hierarchy of Qth-order super nested arrays[1]

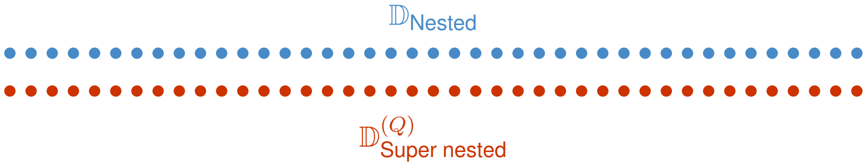
$$\mathbb{S}^{(Q)} = \left(\bigcup_{q=1}^{Q} \mathbb{X}_q^{(Q)} \cup \mathbb{Y}_q^{(Q)}\right) \cup \mathbb{Z}_1^{(Q)} \cup \mathbb{Z}_2^{(Q)},$$





[1] MATLAB routines are available at `http://systems.caltech.edu/dsp/students/clliu/SuperNested.html`

Main properties of super nested arrays: 1) Difference coarray



- $\mathbb{D}^{(Q)}_{\text{Super nested}} = \mathbb{D}_{\text{Nested}}$ if
  - $Q \geq 3$,
  - N_1 and N_2 are sufficiently large.[1]
- Properties of $\mathbb{D}^{(Q)}_{\text{Super nested}}$:
  - Contiguous integers.
  - Hole-free.

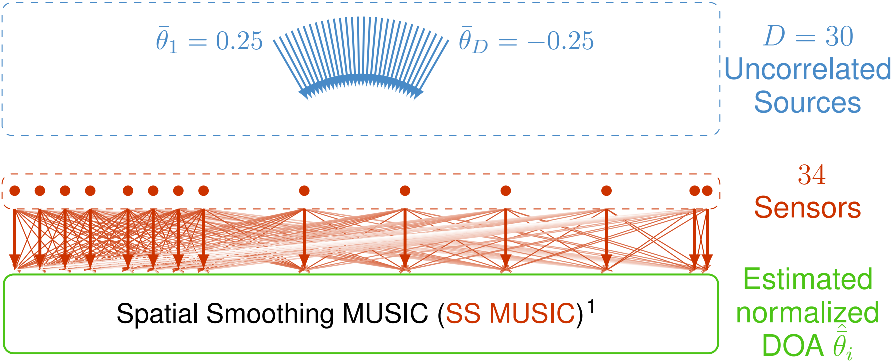
[1] The lower bounds are given in the papers.

Main properties of super nested arrays: 2) Weight functions

	$w(1)$	$w(2)$	$w(3)$
Nested	N_1	$N_1 - 1$	$N_1 - 2$
Super nested	$\begin{cases} 2, & \text{if } N_1 \text{ is even,} \\ 1, & \text{if } N_1 \text{ is odd.} \end{cases}$	$\begin{cases} N_1 - 3, & \text{if } N_1 \text{ is even,} \\ N_1 - 1, & \text{if } N_1 \text{ is odd,} \end{cases}$	$\begin{cases} 3, & \text{if } N_1 = 4, 6, \\ 4, & \text{if } N_1 \text{ is even} \\ & N_1 \geq 8, \\ 1, & \text{if } N_1 \text{ is odd,} \end{cases}$
High order super nested $Q \geq 3$	$\begin{cases} 2, & \text{if } N_1 \text{ is even,} \\ 1, & \text{if } N_1 \text{ is odd,} \end{cases}$	$\begin{cases} 2\lfloor N_1/4 \rfloor + 1, & \text{if } N_1 \text{ is odd,} \\ N_1/2 + 1, & \text{if } N_1 = 8k - 2, \\ N_1/2 - 1, & \text{if } N_1 = 8k + 2, \\ N_1/2, & \text{otherwise,} \end{cases}$	$\begin{cases} 5, & \text{if } N_1 \text{ is even,} \\ 2, & \text{if } N_1 \text{ is odd,} \end{cases}$

Outline

Simulation procedure

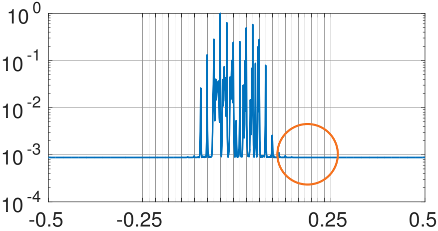


0 dB SNR, 200 snapshots, RMSE $E = \left(\frac{1}{D}\sum_{i=1}^{D}(\hat{\bar{\theta}}_i - \bar{\theta}_i)^2\right)^{1/2}$

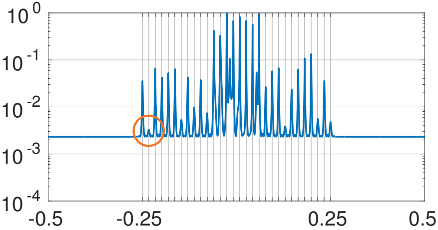
[1] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010; Liu and Vaidyanathan, *IEEE Signal Proc. Lett.*, 2015.

MUSIC spectra (34 sensors, 30 sources)

Nested array ($E = 0.10209$)

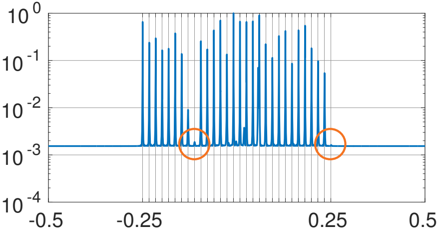


Coprime array ($E = 0.019742$)



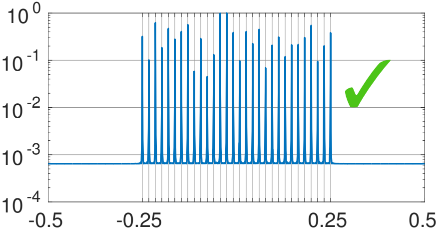
Super nested array

$Q = 2, E = 0.013414$



High order super nested array

$Q = 3, E = 0.00015819$



Outline

Concluding remarks

- High order super nested arrays
  - They have the same number of sensors as (super) nested arrays.
  - They have the same difference coarray as (super) nested arrays if N_1 and N_2 are sufficiently large.
  - They have reduced mutual coupling than (super) nested arrays.
  - They can be constructed recursively from (super) nested arrays.
- In the future, decoupling algorithms will improve the performance.[1]
- For more information, please go to our project website: `http://systems.caltech.edu/dsp/students/clliu/SuperNested.html`

Thank you!

[1] Friedlander and Weiss, *IEEE Trans. Antennas Propag.*, 1991; BouDaher, Ahmad, Amin, and Hoorfar, *EUSIPCO*, 2015.

The data model (ideal)

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}\left(\bar{\theta}_i\right) + \mathbf{n}_{\mathbb{S}},$$

- $\mathbb{S}$: An integer set for the sensor locations, in units of $\lambda/2$.
- $\bar{\theta}_i = (d/\lambda)\sin\theta_i$: the normalized DOA ($-1/2 \leq \bar{\theta}_i < 1/2$).
- A_i: The complex amplitude for the ith source.
- $\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) = [e^{j2\pi\bar{\theta}_i n}]_{n\in\mathbb{S}}$: steering vectors.

Statistical Assumptions

- A_i: zero mean, variance σ_i^2.
- $\mathbf{n}_{\mathbb{S}}$: zero mean, covariance $\sigma^2\mathbf{I}$.
- Sources are uncorrelated: $\mathbb{E}[A_i A_j^*] = \sigma_i^2 \delta_{i,j}$.
- Sources are uncorrelated to the noise: $\mathbb{E}[A_i \mathbf{n}_{\mathbb{S}}^H] = \mathbf{0}$.
- $\bar{\theta}_i$ is considered to be fixed but unknown.

The data model in the presence of mutual coupling[1]

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{C} \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathbb{S}},$$

- $\mathbf{C}$: mutual coupling matrix satisfying

$$\langle \mathbf{C} \rangle_{n_1,n_2} = \begin{cases} c_{|n_1-n_2|}, & \text{if } |n_1 - n_2| \leq B, \\ 0, & \text{otherwise,} \end{cases}$$

- n_1 and n_2 are sensor locations.
- $1 = c_0 > |c_1| > |c_2| > \cdots > |c_B|$.
- In this paper, we assume that $|c_k/c_\ell| = \ell/k$.
- Mutual coupling is a function of sensor separations.

[1] Friedlander and Weiss, *IEEE Trans. Antennas Propag.*, 1991.

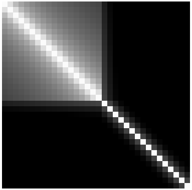
The mutual coupling models in simulations

$$B = 100, \quad c_1 = 0.6e^{j\frac{\pi}{3}}, \quad c_\ell = \frac{c_1}{\ell} e^{-j\frac{\pi}{8}(\ell-1)}, \quad \text{for } \ell = 2, 3, \ldots, B$$

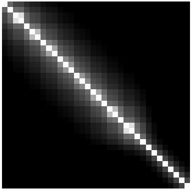
Coefficients	c_1	c_2	c_3	c_4	c_5
Real	0.3000	0.2380	0.1932	0.1487	0.1039
Imaginary	0.5196	0.1826	0.0518	-0.0196	-0.0600

Magnitudes of mutual coupling matrices, $|[\mathbf{C}]_{i,j}|$

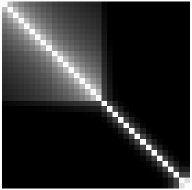
Nested array



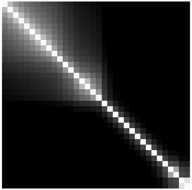
Coprime array



Second-order super nested array



Third-order super nested array



Another mutual coupling model: King's formula[1]

If the sensor array is a linear dipole array, $\mathbf{C}$ can be written as

$$\mathbf{C} = (Z_A + Z_L)(\mathbf{Z} + Z_L\mathbf{I})^{-1},$$

where Z_A and Z_L are the element/load impedance, respectively. $\langle \mathbf{Z} \rangle_{n_1,n_2}$ is given by

$$\begin{cases} \frac{\eta_0}{4\pi}\left(0.5772 + \ln(2\beta l) - \mathrm{Ci}(2\beta l) + j\mathrm{Si}(2\beta l)\right), & \text{if } n_1 = n_2, \\ \frac{\eta_0}{4\pi}\left(\langle \mathfrak{R} \rangle_{n_1,n_2} + j\langle \mathfrak{X} \rangle_{n_1,n_2}\right), & \text{if } n_1 \neq n_2. \end{cases}$$

Here $\eta_0 = \sqrt{\mu_0/\epsilon_0} \approx 120\pi$ is the intrinsic impedance. $\beta = 2\pi/\lambda$ is the wavenumber, where λ is the wavelength. l is the length of dipole antennas. $\mathfrak{R}$ and $\mathfrak{X}$ are

$$\langle \mathfrak{R} \rangle_{n_1,n_2} = \sin(\beta l)\left(-\mathrm{Si}(u_0) + \mathrm{Si}(v_0) + 2\mathrm{Si}(u_1) - 2\mathrm{Si}(v_1)\right)$$

$$+ \cos(\beta l)(\mathrm{Ci}(u_0) + \mathrm{Ci}(v_0) - 2\mathrm{Ci}(u_1) - 2\mathrm{Ci}(v_1) + 2\mathrm{Ci}(\beta d_{n_1,n_2})) - \left(2\mathrm{Ci}(u_1) + 2\mathrm{Ci}(v_1) - 4\mathrm{Ci}(\beta d_{n_1,n_2})\right),$$

$$\langle \mathfrak{X} \rangle_{n_1,n_2} = \sin(\beta l)\left(-\mathrm{Ci}(u_0) + \mathrm{Ci}(v_0) + 2\mathrm{Ci}(u_1) - 2\mathrm{Ci}(v_1)\right)$$

$$+ \cos(\beta l)(-\mathrm{Si}(u_0) - \mathrm{Si}(v_0) + 2\mathrm{Si}(u_1) + 2\mathrm{Si}(v_1) - 2\mathrm{Si}(\beta d_{n_1,n_2})) + \left(2\mathrm{Si}(u_1) + 2\mathrm{Si}(v_1) - 4\mathrm{Si}(\beta d_{n_1,n_2})\right).$$

where $d_{n_1,n_2} = |n_1 - n_2|\lambda/2$ is the distance between sensors. The parameters u_0, v_0, u_1, and v_1 are

$$u_0 = \beta\left(\sqrt{d_{n_1,n_2}^2 + l^2} - l\right), \qquad v_0 = \beta\left(\sqrt{d_{n_1,n_2}^2 + l^2} + l\right),$$

$$u_1 = \beta\left(\sqrt{d_{n_1,n_2}^2 + 0.25l^2} - 0.5l\right), \qquad v_1 = \beta\left(\sqrt{d_{n_1,n_2}^2 + 0.25l^2} + 0.5l\right).$$

Here $\mathrm{Si}(u) = \int_0^u \frac{\sin t}{t} \mathrm{d}t$ and $\mathrm{Ci}(u) = \int_\infty^u \frac{\cos t}{t} \mathrm{d}t$ are sine/cosine integrals.

---

[1] King, *IRE Trans. Antennas Propag.*, 1957.

Properties of the weight functions $w(m)$[1]

The weight function $w(m)$

The number of sensor pairs with separation m.

For any linear array with N sensors, weight functions satisfy

1. $w(0)$ equals the total number of sensors, i.e.,

$$w(0) = N.$$

2. The sum of the weight functions is purely dependent on N.

$$\sum_{m \in \mathbb{D}} w(m) = N^2.$$

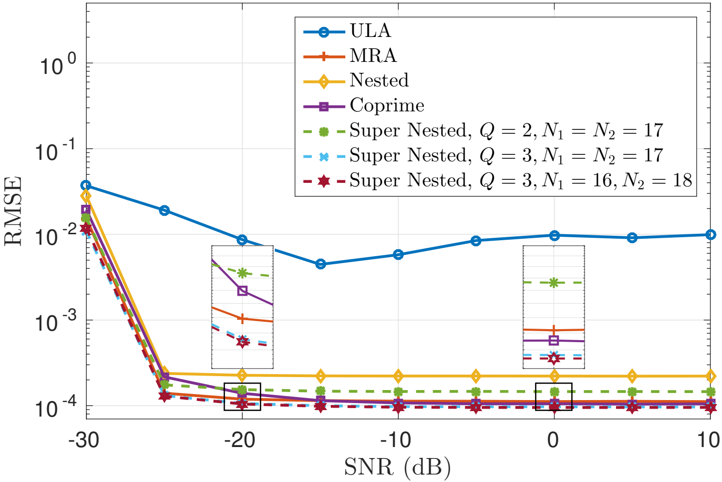
3. Weight functions are symmetric.

$$w(m) = w(-m), \qquad \text{for } m \in \mathbb{D}.$$

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Performance over SNR[1]

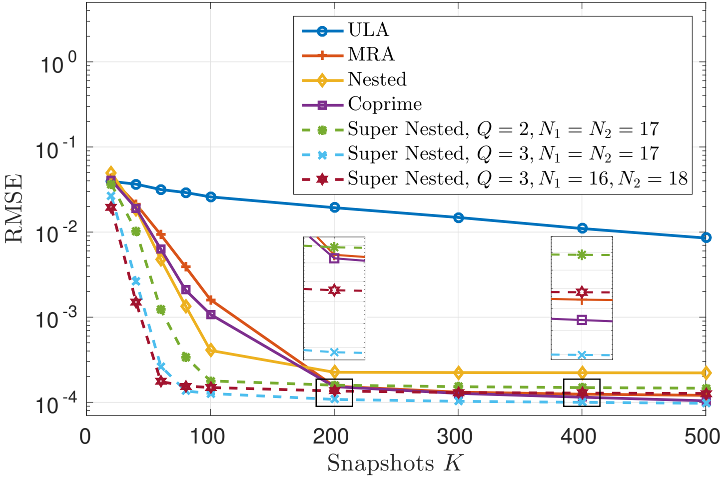
(a)



34 sensors, 20 equal-power sources, 500 snapshots, dipole model, $Z_A = Z_L = 50$, $l = \lambda/2$, $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$, 1000 runs.

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Performance over Snapshots[1]

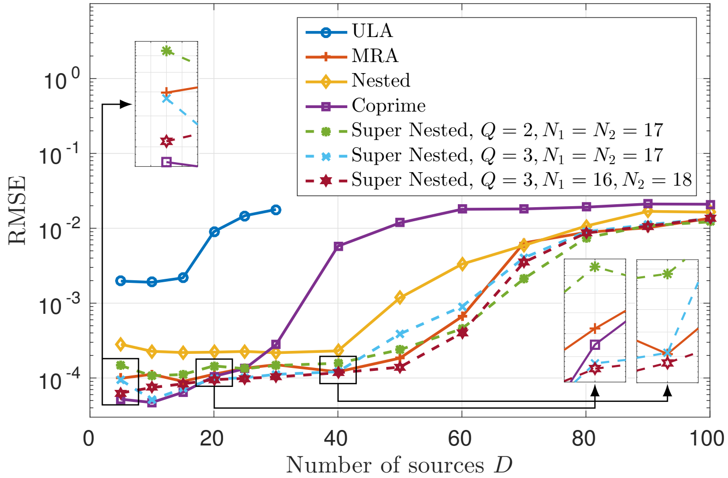
(b) 

34 sensors, 20 equal-power sources, 0dB SNR, dipole model, $Z_A = Z_L = 50$, $l = \lambda/2$, $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$, 1000 runs.

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Performance over Number of sources[1]

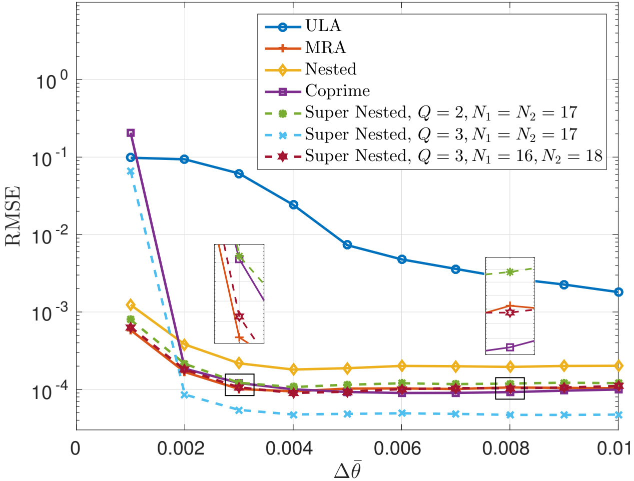
(c)



34 sensors, equal-power sources, 0dB SNR, 500 snapshots, dipole model, $Z_A = Z_L = 50$, $l = \lambda/2$, $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$, 1000 runs.

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Performance over two closely spaced sources[1]

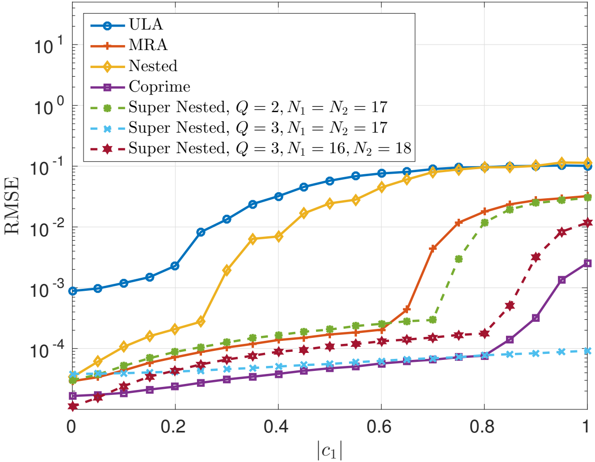


34 sensors, two equal-power sources at $\bar{\theta}_1 = 0.2 + \Delta\bar{\theta}/2, \bar{\theta}_1 = 0.2 - \Delta\bar{\theta}/2,$ 0dB SNR, 500 snapshots, dipole model, $Z_A = Z_L = 50, l = \lambda/2, 1000$ runs.

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Performance over mutual coupling models[1]

(a)

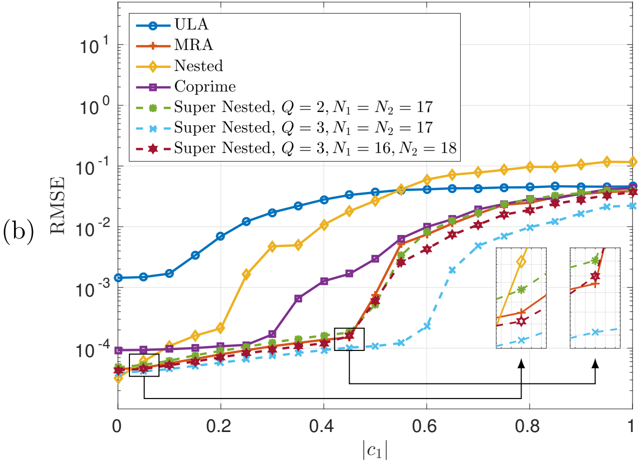


10 sources, 34 sensors

0dB SNR, 500 snapshots, Toeplitz model, phases of c_ℓ are random. $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$, 1000 runs.

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Performance over mutual coupling models[1]



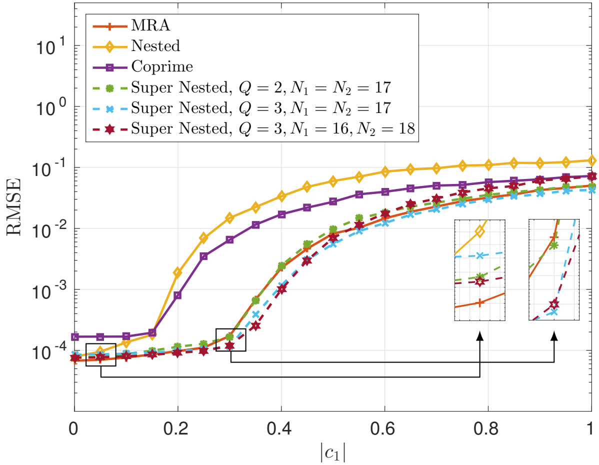
20 sources, 34 sensors

0dB SNR, 500 snapshots, Toeplitz model, phases of c_ℓ are random. $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$, 1000 runs.

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Performance over mutual coupling models[1]

(c)



40 sources, 34 sensors

0dB SNR, 500 snapshots, Toeplitz model, phases of c_ℓ are random. $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$, 1000 runs.

[1] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.