Supplementary Material for "Hourglass Arrays, and Other Novel 2D Sparse Arrays with Reduced Mutual Coupling [1]"

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I. A Numerical Example for Theorem 2 of [1]

In this section, a numerical example will be given to clarify Theorem 2 of [1]. Let us consider a POBA-L with $L = 4$. The array configuration is illustrated in Fig. 1(a), where $\mathfrak{g}_1 = \{1, 3, 5, 7\}$, $\mathfrak{g}_2 = \{2, 4, 6\}$, and

$$\mathfrak{h}_{1,1} = \{1, 3, 4, 8, 12, 13, 15\}, \quad \mathfrak{h}_{1,2} = \{2, 7, 9, 14\}, \tag{1}$$

$$\mathfrak{h}_{1,3} = \{6, 10\}, \quad \mathfrak{h}_{1,4} = \{5, 11\}. \tag{2}$$

Due to Fig. 1(b), it can be seen that $(\pm 6, \pm 11)$ and $(\pm 8, \pm 6)$ are holes in the coarray.

Based on Theorem 2 of [1], the holes of the coarray can be precisely determined by $\mathfrak{h}_{1,\ell}$ for $1 \leq \ell \leq L = 4$. To proceed, the sets $\mathbb{P}_{\ell'}$ for $2 \leq \ell' \leq \ell$ are given by

$$\begin{aligned}\mathbb{P}_2 &= \mathfrak{h}_{1,1} \oplus \mathfrak{h}_{1,1} \\ &= \{2, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, \\ &\quad 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 30\}, \end{aligned} \tag{3}$$

$$\begin{aligned}\mathbb{P}_3 &= \mathfrak{h}_{1,1} \oplus \mathfrak{h}_{1,2} \\ &= \{3, 5, 6, 8, 10, 11, 12, 13, 14, 15, \\ &\quad 17, 18, 19, 20, 21, 22, 24, 26, 27, 29\}, \end{aligned} \tag{4}$$

$$\begin{aligned}\mathbb{P}_4 &= (\mathfrak{h}_{1,1} \oplus \mathfrak{h}_{1,3}) \cup (\mathfrak{h}_{1,2} \oplus \mathfrak{h}_{1,2}) \\ &= \{4, 7, 9, 10, 11, 13, 14, 16, 18, 19, 21, 22, 23, 25, 28\}. \end{aligned} \tag{5}$$

Now we can check (10) in [1]. Comparing $\mathfrak{h}_{1,\ell}$ with $\mathbb{P}_{\ell'}$ for $2 \leq \ell' \leq \ell$ leads to

$$\begin{aligned} &\mathfrak{h}_{1,2} \subseteq \mathbb{P}_2, && \mathfrak{h}_{1,3} \not\subseteq \mathbb{P}_2, && \mathfrak{h}_{1,4} \subseteq \mathbb{P}_2, \\ &\mathfrak{h}_{1,3} \subseteq \mathbb{P}_3, && \mathfrak{h}_{1,4} \subseteq \mathbb{P}_3, && \mathfrak{h}_{1,4} \not\subseteq \mathbb{P}_4. \end{aligned}$$

It can be inferred that $10 \in \mathfrak{h}_{1,3}$ and $10 \notin \mathbb{P}_2$. Therefore, this array does not have hole-free coarrays. Furthermore, due to the necessity proof in Appendix D of [1], the holes can be identified precisely. For $N_x = 9$, $N_y = 17$, $\ell' = 2$, and $n = 10 \in \mathfrak{h}_{1,3}$, the hole is $(N_x + 1 - \ell', N_y - 1 - n) = (8, 6)$, which is exactly shown in Fig. 1(b).

The importance of Theorem 2 of [1] is that, we can enumerate all the holes in the coarray by analyzing $\mathfrak{h}_{1,\ell}$ and $\mathbb{P}_{\ell'}$. Based on (1) to (5), it can be deduced that

$$\begin{aligned} &5 \in \mathfrak{h}_{1,4}, && 5 \notin \mathbb{P}_4, && \text{Hole: } (6, 11), \\ &6 \in \mathfrak{h}_{1,3}, && 6 \notin \mathbb{P}_2 - (N_y - 1), && \text{Hole: } (8, -6), \\ &11 \in \mathfrak{h}_{1,4}, && 11 \notin \mathbb{P}_4 - (N_y - 1), && \text{Hole: } (6, -11). \end{aligned}$$

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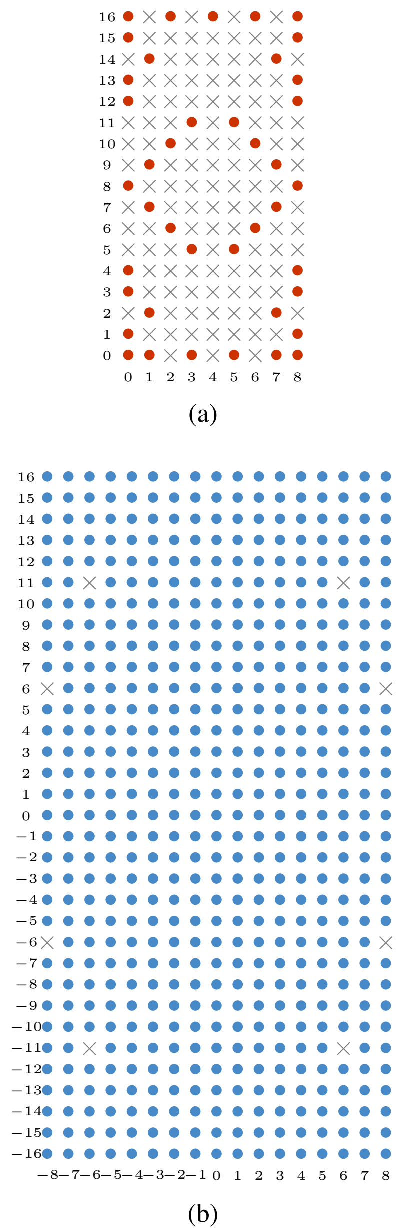


Fig. 1. A numerical example for Theorem 2 of [1]. (a) The physical array of a POBA-L with $L = 4$, $N_x = 9$, and $N_y = 17$. (b) The corresponding difference coarray $\mathbb{D}_{\text{POBA-}L}$. Bullets represent the elements in the physical array or in the coarray while crosses denote empty space.

The remaining four holes in Fig. 1(b) can be obtained by using the symmetry of the weight function $w(\mathbf{m}) = w(-\mathbf{m})$ [2].

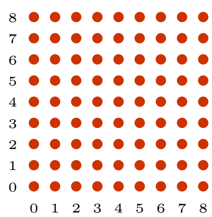
II. Array Configurations in Section VIII of [1]

Figs. 2 and 3 show the array geometries of the planar arrays in Section VIII of [1]. The total number of sensors is 81 for each array. It can be seen from Figs. 2 and 3 that the weight functions $w(0,1), w(1,0), w(1,1), w(1,-1)$ are identical to those given in Table I of [1].

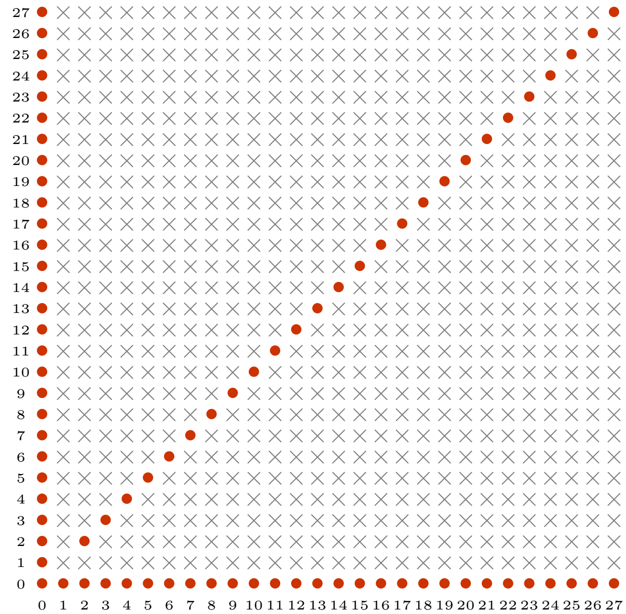
References

[1] C.-L. Liu and P. P. Vaidyanathan, "Hourglass arrays, and other novel 2D sparse arrays with reduced mutual coupling," *submitted to IEEE Trans. Signal Process.*, 2016.

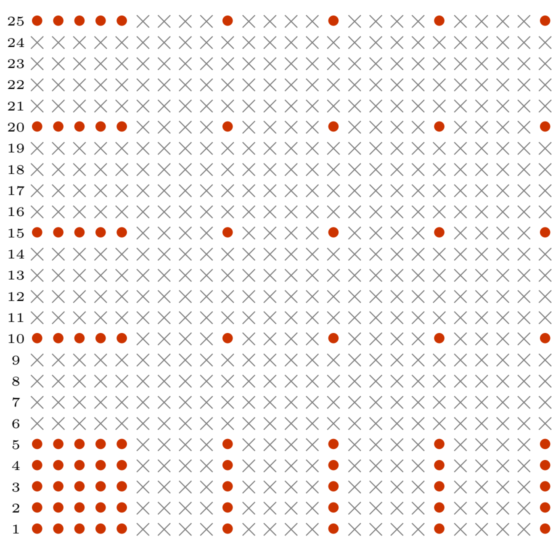
[2] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug 2010.



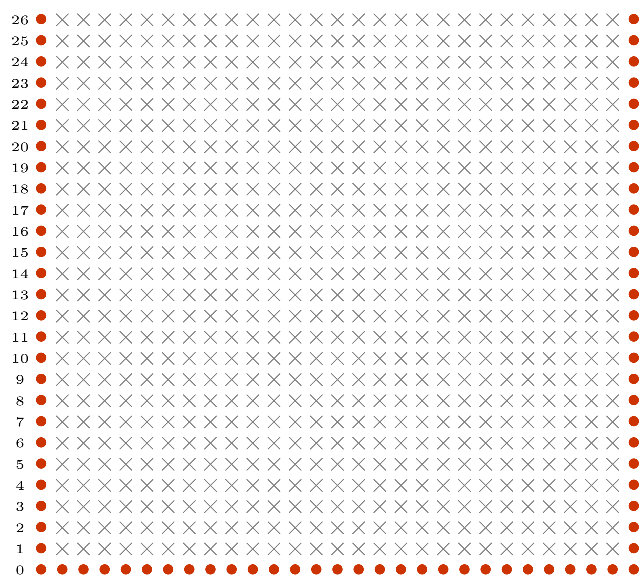
(a) Uniform rectangular array, $N_x = N_y = 9$.
The number of sensors is 81.
$w(1,0) = 72,\ w(0,1) = 72,$
$w(1,1) = 64,\ w(1,-1) = 64.$



(b) Billboard array, $N_x = N_y = 28$.
The number of sensors is 81.
$w(1,0) = 27,\ w(0,1) = 27,$
$w(1,1) = 25,\ w(1,-1) = 1.$

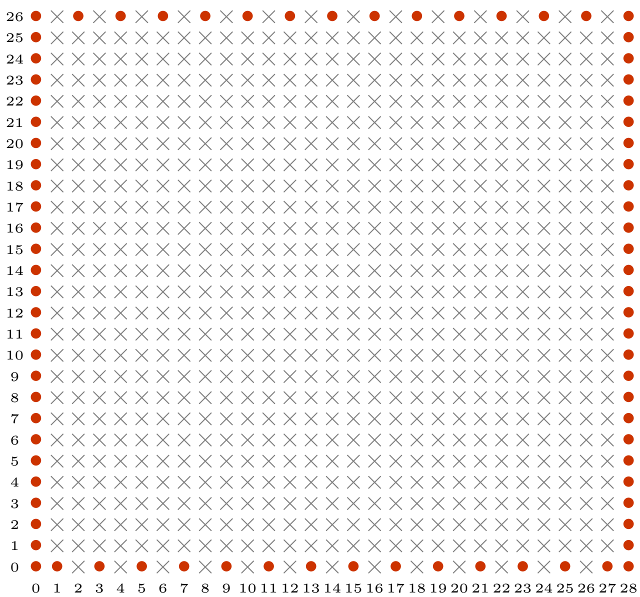


(c) 2D nested array, $N_1 = 4, N_2 = 5$.
The number of sensors is 81.
$w(1,0) = 36,\ w(0,1) = 36,$
$w(1,1) = 16,\ w(1,-1) = 16.$

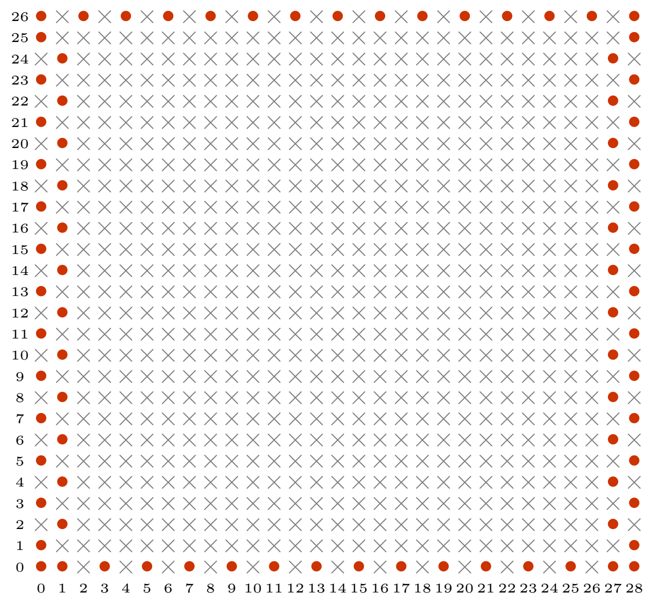


(d) Open box array, $N_x = 29, N_y = 27$.
The number of sensors is 81.
$w(1,0) = 28,\ w(0,1) = 52,$
$w(1,1) = 1,\ w(1,-1) = 1.$

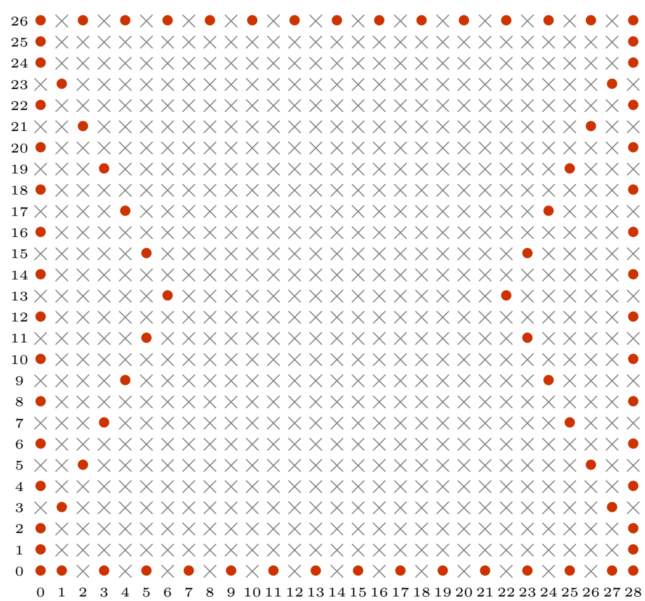
Fig. 2. The geometries for (a) URA, (b) billboard array, (c) 2D nested array, and (c) OBA in Section VIII of [1].



(a) Half open box array, $N_x = 29, N_y = 27$.
The number of sensors is 81.
$w(1,0) = 2,\ w(0,1) = 52,$
$w(1,1) = 1,\ w(1,-1) = 1.$



(b) Half open box array with two layers, $N_x = 29, N_y = 27$.
The number of sensors is 81.
$w(1,0) = 2,\ w(0,1) = 4,$
$w(1,1) = 25,\ w(1,-1) = 25.$



(c) Hourglass array, $N_x = 29, N_y = 27$.
The number of sensors is 81.
$w(1,0) = 2,\ w(0,1) = 8,$
$w(1,1) = 5,\ w(1,-1) = 5.$

Fig. 3. The geometries for (a) HOBA, (b) HOBA-2, and (c) hourglass arrays in Section VIII of [1].