Tensor MUSIC in Multidimensional Sparse Arrays

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***Abstract*—Tensor-based MUSIC algorithms have been successfully applied to parameter estimation in array processing. In this paper, we apply these for sparse arrays, such as nested arrays and coprime arrays, which are known to boost the degrees of freedom to $O(N^2)$ given $O(N)$ sensors. We consider two tensor decomposition methods: CANDECOMP/PARAFAC (CP) and high-order singular value decomposition (HOSVD) to derive novel tensor MUSIC spectra for sparse arrays. It will be demonstrated that the tensor MUSIC spectrum via HOSVD suffers from cross-term issues while the tensor MUSIC spectrum via CP identifies sources unambiguously, even in high-dimensional tensors.**[1]

***Index Terms*—Sparse arrays, MUSIC algorithm, CANDECOMP/PARAFAC (CP), high-order singular value decomposition (HOSVD).**

I. Introduction

A tensor is a high-dimensional array [1], [2]. It enables us to embed the underlying signal characteristics in different dimensions, rather than stacking the measurement into a huge vector. Tensors find application in sensor array processing [3], parameter estimation [4], harmonic retrieval in MIMO radar [5], and vector-sensor array processing [6].

The spirit in tensor-based signal processing is built on the tensor decomposition [1], [2]. Known methods can be divided into two categories: CANDECOMP/PARAFAC (CP) [7], [8] and high-order singular value decomposition (HOSVD) [9]. CP decomposes a given tensor into a sum of rank-one tensors. HOSVD finds a set of unitary matrices across different dimensions that simplifies the original tensor. Both methods reveal the signal characteristics in the rank-one tensors of CP or in the unitary matrices of HOSVD. Therefore, a variety of tensor-based parameter estimation approaches, such as tensor-ESPRIT and unitary tensor-ESPRIT [4], as well as tensor MUSIC [6], [10] have been shown to outperform the associated matrix-based approach.

Sparse arrays such as minimum redundancy arrays [11], nested arrays [12], and coprime arrays [13], provide an enhanced degree of freedom of $O(N^2)$ given $O(N)$ sensors. These find application in DOA estimation [14], sampling [12], [13], space-time adaptive processing [15], and vector-sensor array processing [6].

In this paper, sparse arrays are incorporated with high-dimensional tensor models and the associated tensor MUSIC spectra are proposed. We will later show that HOSVD gives rise to undesirable cross-terms in the tensor MUSIC spectrum. On the other hand, if we utilize CP to derive a tensor MUSIC spectrum, cross-terms can be resolved.

The rest of this paper is organized as follows. In Section II, we list the notaions in this paper and then review the main idea of sparse array processing as well as the HOSVD tensor MUSIC spectrum. In Section III-A, we show that sparse arrays described by tensor models can be processed without spatial smoothing, similar to [16]. Then in Section II-C, two tensor MUSIC spectra, based on HOSVD and CP, will be proposed and studied. Some numerical examples will be demonstrated in Section IV while Section V concludes this paper.

II. Preliminaries

A. Notations

Scalars are denoted as italic lowercase letters, column vectors are in bold-face lowercase letters, and bold-face uppercase letters denote matrices. Tensors are written as bold-face calligraphic letters, like $\boldsymbol{\mathcal{A}}$. Sets are denoted by blackboard boldface. $\mathbf{A}^*$, $\mathbf{A}^T$, and $\mathbf{A}^H$ represent the conjugate, the transpose, and the conjugate transpose of a matrix $\mathbf{A}$, respectively. For a real set $\mathbb{A}$, $\mathbb{A}^+$ is a set containing the non-negative part of $\mathbb{A}$. In the following definitions, the tensors $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{C}} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ and $\boldsymbol{\mathcal{B}} \in \mathbb{C}^{J_1\times\cdots\times J_M}$. $[\boldsymbol{\mathcal{A}}]_{i_1,\ldots,i_N}$ extracts the entry of $\boldsymbol{\mathcal{A}}$ indexed by $(i_1,\ldots,i_N)$. All tensor notations are summarized in Appendix.

Given a support set $\mathbb{S}$, the signal defined on it is denoted as a column vector $\mathbf{x}_\mathbb{S}$. We use $[\mathbf{x}_\mathbb{S}]_i$ to denote the i^{th} component of this vector. For $n \in \mathbb{S}$ we use the triangular bracket notation $\langle \mathbf{x}_\mathbb{S}\rangle_n$ to denote the value of the signal at the support location n. For example, if $\mathbb{S} = \{1,3,7\}$ and $\mathbf{x}_\mathbb{S} = [0.1, 0.2, 0.3]^T$, then $[\mathbf{x}_\mathbb{S}]_1 = 0.1, [\mathbf{x}_\mathbb{S}]_2 = 0.2, [\mathbf{x}_\mathbb{S}]_3 = 0.3$, whereas $\langle\mathbf{x}_\mathbb{S}\rangle_1 = 0.1, \langle\mathbf{x}_\mathbb{S}\rangle_3 = 0.2, \langle\mathbf{x}_\mathbb{S}\rangle_7 = 0.3$. These notations extend to matrices: $[\mathbf{R}_\mathbb{S}]_{i_1,i_2} = \mathbb{E}\left[[\mathbf{x}_\mathbb{S}]_{i_1}[\mathbf{x}_\mathbb{S}]^*_{i_2}\right]$ and $\langle\mathbf{R}_\mathbb{S}\rangle_{n_1,n_2} = \mathbb{E}\left[\langle\mathbf{x}_\mathbb{S}\rangle_{n_1}\langle\mathbf{x}_\mathbb{S}\rangle^*_{n_2}\right]$. The same definition works for the tensor case.

B. Sparse Array Processing

Consider a linear sensor array whose sensors are located at nd, where $d = \lambda/2$ and n belongs to an integer set $\mathbb{S}$. A sensor array is said to be *sparse* if some of the adjacent elements have spacing greater than $\lambda/2$. It is known that sparse arrays such as nested arrays [12], coprime arrays [13], [17], and minimum redundancy arrays [11] achieve higher degrees of freedom than the conventional ULA with the same number of sensors.

Assume that D monochromatic waves impinge on the sensor array characterized by $\mathbb{S}$. The received signals on $\mathbb{S}$

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can be modeled as a column vector $\mathbf{x}_{\mathbb{S}}$:

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}\left(\bar{\theta}_i\right) + \mathbf{n}_{\mathbb{S}}. \tag{1}$$

Here A_i and $\bar{\theta}_i$ are the complex amplitude, and the normalized DOA of the i^{th} source where $i = 1, \ldots, D$. $\bar{\theta}_i \in [-\pi/2, \pi/2)$. The *steering vectors* $\mathbf{v}_{\mathbb{S}}\left(\bar{\theta}_i\right)$ are column vectors with entries $e^{j2\pi\bar{\theta}_i n}$ for $n \in \mathbb{S}$. $\mathbf{n}_{\mathbb{S}}$ is an additive noise vector. Note that $\{A_i\}_{i=1}^{D}$ and $\mathbf{n}_{\mathbb{S}}$ are treated as zero-mean random vectors with $\mathbb{E}\left[A_i A_j^*\right] = \sigma_i^2 \delta_{i,j}$, $\mathbb{E}\left[\mathbf{n}_{\mathbb{S}}\mathbf{n}_{\mathbb{S}}^H\right] = \sigma^2 \mathbf{I}$. The sources and the noise are assumed to be uncorrelated. Then, the covariance matrix of $\mathbf{x}_{\mathbb{S}}$ becomes

$$\mathbf{R}_{\mathbb{S}} = \mathbb{E}\left[\mathbf{x}_{\mathbb{S}}\mathbf{x}_{\mathbb{S}}^H\right] = \sum_{i=1}^{D} \sigma_i^2 \mathbf{v}_{\mathbb{S}}\left(\bar{\theta}_i\right)\mathbf{v}_{\mathbb{S}}^H\left(\bar{\theta}_i\right) + \sigma^2 \mathbf{I}. \tag{2}$$

We define the difference coarray $\mathbb{D}$ to be a set that collects every possible difference within $\mathbb{S}$, or equivalently $\mathbb{D} = \{n_1 - n_2 | n_1, n_2 \in \mathbb{S}\}$. Following the details in [12], [14], [16], we rearrange the entries of $\mathbf{R}_{\mathbb{S}}$ into

$$\mathbf{x}_{\mathbb{D}} = \sum_{i=1}^{D} \sigma_i^2 \mathbf{v}_{\mathbb{D}}\left(\bar{\theta}_i\right) + \sigma^2 \mathbf{e}_0, \tag{3}$$

where $\langle \mathbf{e}_0 \rangle_m = \delta_{m,0}$. In the context of sparse arrays, $\mathbb{S}$ is designed properly so that its difference coarray $\mathbb{D}$ possesses a long ULA section in the middle [11]–[13], [17]. This ULA portion is called $\mathbb{U}$ in the following development. The spatial smoothing [12], [14] is applied to $\mathbf{x}_{\mathbb{U}}$ to construct a full-rank matrix $\mathbf{R}_{ss}$, on which the MUSIC algorithm works.

Apart from implementing spatial smoothing, in [16], the authors proposed another matrix $\widetilde{\mathbf{R}}$ replacing the role of the spatially smoothed matrix $\widetilde{\mathbf{R}}_{ss}$. It was shown in [16] that $\widetilde{\mathbf{R}}$ follows the same formulation as the augmented covariance matrix method [18]. This method not only conceptually simplifies the analysis but also reduces the overall computational complexity, compared to the spatially smoothed matrix. Thus in this paper, we will adopt the formulation in [16] and generalize it into the tensor case.

C. Tensor MUSIC Spectrum Based on High-Order SVD

Apart from the one-dimensional DOA estimation using linear arrays, more sophisticated designs exist to extract more information. For instance, planar arrays detect elevation/azimuth information jointly, vector-sensor arrays resolve DOA/polarization, and space-time processing estimates DOA/Doppler profiles. It is natural to model these characteristics as a data tensor, which is analogous to $\mathbf{x}_{\mathbb{S}}$ in (1).

Throughout the paper, an R-dimensional normalized harmonic parameter $\bar{\boldsymbol{\mu}} = [\bar{\mu}^{(1)}, \bar{\mu}^{(2)}, \ldots, \bar{\mu}^{(R)}]^T \in [-1/2, 1/2)^R$ will be considered. In practice, $\bar{\mu}^{(r)}$ can be viewed as normalized DOA, normalized Doppler, or polarization, depending on the problem of interest. Next, a collection of integer sets $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_R$ identifies the array configuration (if its physical model corresponds to sensors) or the sampling pattern (if it corresponds to time domain sampling) over different dimensions. We write the associated difference sets to be $\mathbb{D}_1, \mathbb{D}_2, \ldots, \mathbb{D}_R$, respectively. Then, the tensor version of (1) becomes

$$\boldsymbol{\mathcal{X}}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \boldsymbol{\mathcal{V}}_{\mathbb{S}}\left(\bar{\boldsymbol{\mu}}\right) + \boldsymbol{\mathcal{N}}_{\mathbb{S}}. \tag{4}$$

Here $\boldsymbol{\mathcal{X}}_{\mathbb{S}}$ is a data tensor over $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_R$ and $\boldsymbol{\mathcal{N}}_{\mathbb{S}}$ is an additive noise tensor. $\boldsymbol{\mathcal{V}}_{\mathbb{S}}\left(\bar{\boldsymbol{\mu}}\right)$ is the R-dimensional array steering tensor defined by

$$\boldsymbol{\mathcal{V}}_{\mathbb{S}}\left(\bar{\boldsymbol{\mu}}\right) = \mathbf{v}_{\mathbb{S}_1}\left(\bar{\mu}_i^{(1)}\right) \circ \mathbf{v}_{\mathbb{S}_2}\left(\bar{\mu}_i^{(2)}\right) \circ \cdots \circ \mathbf{v}_{\mathbb{S}_R}\left(\bar{\mu}_i^{(R)}\right),$$

where the steering vectors $\mathbf{v}_{\mathbb{S}_r}\left(\bar{\mu}_i^{(r)}\right)$ follows the same definition as that in (1). By assuming the same statistical properties as (1), the *covariance tensor* (or interspectral tensor) becomes

$$\boldsymbol{\mathcal{R}}_{\mathbb{S}} = \mathbb{E}\left[\boldsymbol{\mathcal{X}}_{\mathbb{S}} \circ \boldsymbol{\mathcal{X}}_{\mathbb{S}}^*\right] = \sum_{i=1}^{D} \sigma_i^2 \boldsymbol{\mathcal{V}}_{\mathbb{S}}\left(\bar{\boldsymbol{\mu}}\right) \circ \boldsymbol{\mathcal{V}}_{\mathbb{S}}^*\left(\bar{\boldsymbol{\mu}}\right) + \sigma^2 \boldsymbol{\mathcal{I}}, \tag{5}$$

where $\left[\boldsymbol{\mathcal{I}}\right]_{i_1, i_2, \ldots, i_R, i_1', i_2', \ldots, i_R'} = \prod_{r=1}^{R} \delta_{i_r, i_r'}$.

Recently, in [10], a tensor MUSIC algorithm was proposed based on the HOSVD of the covariance tensor[2]:

$$\begin{aligned}\boldsymbol{\mathcal{R}}_{\mathbb{S}} = \boldsymbol{\mathcal{K}} &\times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \cdots \times_R \mathbf{U}_R \\ &\times_{R+1} \mathbf{U}_1^* \times_{R+1} \mathbf{U}_2^* \times_{R+1} \cdots \times_{2R} \mathbf{U}_R^*.\end{aligned}$$

Here $\boldsymbol{\mathcal{R}}_{\mathbb{S}}, \boldsymbol{\mathcal{K}} \in \mathbb{C}^{|\mathbb{S}_1| \times \cdots \times |\mathbb{S}_R| \times |\mathbb{S}_1| \times \cdots \times |\mathbb{S}_R|}$ and $\mathbf{U}_r$ is a unitary matrix of size $|\mathbb{S}_r| \times |\mathbb{S}_r|$ for $r = 1, 2, \ldots, R$. To compute the MUSIC spectrum, the signal/noise subspaces are separated based on the number of sources D. Writing the bases of signal/noise subspaces on the r^{th} dimension as matrices $\mathbf{U}_{r,s}$ and $\mathbf{U}_{r,n}$, respectively, the tensor MUSIC spectrum is defined as

$$\begin{aligned}&P_{ULA,HOSVD}\left(\bar{\boldsymbol{\mu}}\right) \\ &= \left\| \boldsymbol{\mathcal{V}}_{\mathbb{S}}\left(\bar{\boldsymbol{\mu}}\right) \times_1 \mathbf{U}_{1,n}\mathbf{U}_{1,n}^H \times_2 \cdots \times_R \mathbf{U}_{R,n}\mathbf{U}_{R,n}^H \right\|_F^{-2}. \end{aligned} \tag{6}$$

Once the tensor MUSIC spectrum $P_{ULA,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ is obtained, the source parameters can be estimated by locating prominent peaks in the tensor MUSIC spectrum.

III. Proposed Tensor MUSIC Spectrum

In this section, we will first formulate $\widetilde{\boldsymbol{\mathcal{R}}}$, which is a tensor version of $\widetilde{\mathbf{R}}$ in [16] and then propose an alternative to the MUSIC spectrum of [6], [10]. We will theoretically prove that there are some cases where the tensor MUSIC spectrum according to HOSVD [10] cannot resolve sources but our proposed method can distinguish such source distributions.

A. Formulation of $\widetilde{\boldsymbol{\mathcal{R}}}$

In the tensor model, given K snapshots $\widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{S}}(k)$ for $k = 1, 2, \ldots, K$, the covariance tensor of $\boldsymbol{\mathcal{X}}_{\mathbb{S}}$ can be estimated as

$$\widetilde{\boldsymbol{\mathcal{R}}}_{\mathbb{S}} = \frac{1}{K} \sum_{k=1}^{K} \widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{S}}(k) \circ \widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{S}}^*(k),$$

[2]This is a generalization of the tensor MUSIC algorithm in [10], where the special case of DOA/polarization was considered.

which serves as a finite-snapshot version of (5). We now define the finite-snapshot, tensor version of (3), denoted as $\widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{D}}$. First we define a set $\mathbb{M}(m_1, m_2, \ldots, m_R)$ as

$$\begin{aligned}&\mathbb{M}(m_1, m_2, \ldots, m_R)\\&= \{(n_1, n_2, \ldots, n_R, n'_1, n'_2, \ldots, n'_R)\\&\qquad \mid n_r, n'_r \in \mathbb{S}_r,\ n_r - n'_r = m_r,\ r = 1, 2, \ldots, R\},\end{aligned}$$

with $m_r \in \mathbb{D}_r$ for $r = 1, 2, \ldots, R$. Then the tensor defined over the difference coarray $\widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{D}}$ satisfies

$$\langle \widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{D}} \rangle_{m_1, m_2, \ldots, m_R} = \sum \frac{\langle \widetilde{\boldsymbol{\mathcal{R}}}_{\mathbb{S}} \rangle_{n_1, n_2, \ldots, n_R, n'_1, n'_2, \ldots, n'_R}}{|\mathbb{M}(m_1, m_2, \ldots, m_R)|},$$

with $(n_1, n_2, \ldots, n_R, n'_1, n'_2, \ldots, n'_R) \in \mathbb{M}(m_1, m_2, \ldots, m_R)$ in the summation.

Finally, $\widetilde{\boldsymbol{\mathcal{R}}}$ can be formulated as a tensor by writing

$$\langle \widetilde{\boldsymbol{\mathcal{R}}} \rangle_{p_1, p_2, \ldots, p_R, p'_1, p'_2, \ldots, p'_R} = \langle \widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{D}} \rangle_{m_1, m_2, \ldots, m_R}, \tag{7}$$

where $p_r, p'_r \in \mathbb{U}_r^+$ and $p_r - p'_r = m_r \in \mathbb{U}_r$ for $r = 1, 2, \ldots, R$ and $\mathbb{U}_r^+$ is the non-negative part of the ULA section of $\mathbb{D}_r$.

As a example, consider $R = 2$, $\mathbb{S}_1 = \{1, 2, 3, 6\}$ (nested array with $N_1 = N_2 = 2$) and $\mathbb{S}_2 = \{0, 2, 4, 3, 6, 9\}$ (coprime array with $M = 2, N = 3$), we obtain the difference sets $\mathbb{D}_1 = \{-5, -4, \ldots, 4, 5\}$ and $\mathbb{D}_2 = \{-9, -7, -6, \ldots, 6, 7, 9\}$. The non-negative parts of the ULA section of $\mathbb{D}_1$ and $\mathbb{D}_2$ are $\mathbb{U}_1^+ = \{0, 1, \ldots, 5\}$ and $\mathbb{U}_2^+ = \{0, 1, \ldots, 7\}$, respectively. The cardinalities of these sets are $|\mathbb{S}_1| = 4$, $|\mathbb{S}_2| = 6$, $|\mathbb{D}_1| = 11$, $|\mathbb{D}_2| = 17$, $|\mathbb{U}_1^+| = 6$, and $|\mathbb{U}_2^+| = 8$. In such configuration, $\widetilde{\boldsymbol{\mathcal{R}}}_{\mathbb{S}} \in \mathbb{C}^{4\times 6\times 4\times 6}$, $\widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{D}} \in \mathbb{C}^{11\times 17}$, and $\widetilde{\boldsymbol{\mathcal{R}}} \in \mathbb{C}^{6\times 8\times 6\times 8}$.

Here are some remarks on $\widetilde{\boldsymbol{\mathcal{R}}}$:

1) $\widetilde{\boldsymbol{\mathcal{R}}}$ avoids implementing the spatial smoothing step in tensors, which is quite complicated especially when R is large. According to (7), all we need to do is to reshape the R^{th}-order tensor $\widetilde{\boldsymbol{\mathcal{X}}}_{\mathbb{D}}$ into the $2R^{\text{th}}$-order tensor $\widetilde{\boldsymbol{\mathcal{R}}}$. Compared to spatial smoothing, extra multiplications can be saved.
2) The coprime JADE in [15] is actually a special case of our tensor model when $R = 2$. We formulated the space-time processing using Kronecker products to define space-time steering vectors on the coarray-lag domain. By appropriately rearranging the data, the coprime JADE model is seen to be equivalent to our tensor model for $R = 2$. However, in this paper, the tensor model can be generalized for larger R. In addition, the spatial smoothing step in [15] can also be avoided.
3) Assuming the number of snapshots approaches infinity, we obtain

$$\lim_{K\to\infty} \widetilde{\boldsymbol{\mathcal{R}}} = \sum_{i=1}^{D} \sigma_i^2 \boldsymbol{\mathcal{V}}_{\mathbb{U}^+}(\bar{\boldsymbol{\mu}}) \circ \boldsymbol{\mathcal{V}}^*_{\mathbb{U}^+}(\bar{\boldsymbol{\mu}}) + \sigma^2 \boldsymbol{\mathcal{I}}, \tag{8}$$

which owns a similar structure as that in (5). Hence, according to [10], tensor decomposition on $\widetilde{\boldsymbol{\mathcal{R}}}$ enables us to define tensor MUSIC spectra, which will be discussed in the following subsection.

B. Tensor MUSIC Spectrum for Sparse Arrays

In this subsection, two tensor MUSIC spectra for sparse arrays will be proposed. One is based on HOSVD while the other is built on CP. It will be shown that the MUSIC spectrum based on HOSVD is actually a separable spectrum so that it cannot resolve non-separable sources distributions. On the other hand, the one based on CP can distinguish non-separable source profiles.

Definition 1 (Tensor MUSIC for sparse array using HOSVD)**.** *Let the high-order SVD of $\widetilde{\boldsymbol{\mathcal{R}}}$ be*

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{R}}} = \widetilde{\boldsymbol{\mathcal{K}}} &\times_1 \widetilde{\mathbf{U}}_1 \times_2 \widetilde{\mathbf{U}}_2 \times_3 \cdots \times_R \widetilde{\mathbf{U}}_R\\&\times_{R+1} \widetilde{\mathbf{U}}_1^* \times_{R+2} \widetilde{\mathbf{U}}_2^* \times_{R+3} \cdots \times_{2R} \widetilde{\mathbf{U}}_R^*,\end{aligned}$$

where unitary matrices $\widetilde{\mathbf{U}}_r \in \mathbb{C}^{|\mathbb{U}_r^+|\times|\mathbb{U}_r^+|}$. Splitting $\widetilde{\mathbf{U}}_r$ into the signal/noise subspaces, denoted by $\widetilde{\mathbf{U}}_{r,s}$ and $\widetilde{\mathbf{U}}_{r,n}$, respectively, one can define a tensor MUSIC spectrum by

$$\begin{aligned}&P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}})\\&= \left\| \boldsymbol{\mathcal{V}}_{\mathbb{U}^+}(\bar{\boldsymbol{\mu}}) \times_1 \widetilde{\mathbf{U}}_{1,n}\widetilde{\mathbf{U}}_{1,n}^H \times_2 \cdots \times_R \widetilde{\mathbf{U}}_{R,n}\widetilde{\mathbf{U}}_{R,n}^H \right\|_F^{-2}.\end{aligned} \tag{9}$$

Theorem 1. *$P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}})$ is a product of R separable MUSIC spectra. That is,*

$$P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}}) = \prod_{r=1}^{R} \left\| \widetilde{\mathbf{U}}_{r,n}^H \mathbf{v}_{\mathbb{U}_r^+}\left(\bar{\mu}^{(r)}\right) \right\|_2^{-2}.$$

Proof. From Definition 1, we can simplify the tensor into

$$\left(\widetilde{\mathbf{U}}_{1,n}\widetilde{\mathbf{U}}_{1,n}^H \mathbf{v}_{\mathbb{U}_1^+}\left(\bar{\mu}^{(1)}\right)\right) \circ \cdots \circ \left(\widetilde{\mathbf{U}}_{R,n}\widetilde{\mathbf{U}}_{R,n}^H \mathbf{v}_{\mathbb{U}_1^+}\left(\bar{\mu}^{(R)}\right)\right).$$

Taking the Frobenius norm of the above expression gives $P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}}) = \prod_{r=1}^{R} \left\| \widetilde{\mathbf{U}}_{r,n}\widetilde{\mathbf{U}}_{r,n}^H \mathbf{v}_{\mathbb{U}_r^+}\left(\bar{\mu}^{(r)}\right) \right\|_2^{-2}$. Since ℓ_2-norm is unitarily invariant, dropping the $\widetilde{\mathbf{U}}_{r,n}$ terms completes the proof. □

According to Theorem 1, $P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}})$ generates undesirable cross terms in the parameter space. For instance, assume that there are $D = 2$ sources and the order R of $\boldsymbol{\mathcal{X}}_{\mathbb{S}}$ is 2. The actual source distribution is $\bar{\boldsymbol{\mu}}_1 = (0, 0.1)$ and $\bar{\boldsymbol{\mu}}_2 = (-0.2, -0.1)$. If infinite-snapshots are available and signal/noise subspaces are determined exactly, $P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}})$ has poles at $\bar{\boldsymbol{\mu}}_1$ and $\bar{\boldsymbol{\mu}}_2$. In addition, however, the cross terms $\bar{\boldsymbol{\mu}}_{12} = (0, -0.1)$ and $\bar{\boldsymbol{\mu}}_{21} = (-0.2, 0.1)$ are also poles of $P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}})$. Thus, by looking at $P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}})$, we will infer that there could be *four* sources with parameter $\bar{\boldsymbol{\mu}}_1$, $\bar{\boldsymbol{\mu}}_2$, $\bar{\boldsymbol{\mu}}_{12}$, and $\bar{\boldsymbol{\mu}}_{21}$, respectively, causing ambiguity in this tensor MUSIC spectrum.

A intuitive answer as to why cross terms come into picture is as follows. If we match the tensor model (5) with the signal/noise subspaces, we know that $\mathbf{v}_{\mathbb{U}_r^+}\left(\bar{\mu}_i^{(r)}\right)$ belongs to the column space of $\mathbf{U}_{r,s}$ for $i = 1, 2, \ldots, D$. Therefore, $\mathbf{U}_{r,n}^H \mathbf{v}_{\mathbb{U}_r^+}\left(\bar{\mu}_i^{(r)}\right) = 0$. According to Theorem 1, $P_{Sparse,HOSVD}(\bar{\boldsymbol{\mu}}_i)$ goes to infinity. However, this approach produces a separate MUSIC spectrum, as Theorem 1 states. It ends up estimating parameters separately in different dimensions without considering the connection among them.

Surely there must exist a joint estimation approach to find pairs of parameters in multiple dimensions. For instance, in [4], tensor ESPRIT is built upon a joint Schur decomposition or a simultaneous diagonalization algorithm. We need to seek another definition of tensor MUSIC spectrum to incorporate pairing information provided by our tensor model.

It can be observed from (8) that $\widetilde{\mathcal{R}}$ contains a low-rank component, which is constructed from D rank-one tensors. These components characterize a signal subspace, which enables us to define a MUSIC spectrum. This idea can be summarized as follows:

Definition 2 (Tensor MUSIC for sparse array using CP)**.** *Consider the CP decomposition of $\widetilde{\mathcal{R}}$,*

$$\widetilde{\mathcal{R}} = \sum_{\ell=1}^{D} \widetilde{\mathbf{a}}_\ell^{(1)} \circ \widetilde{\mathbf{a}}_\ell^{(2)} \circ \cdots \circ \widetilde{\mathbf{a}}_\ell^{(R)} \circ \widetilde{\mathbf{a}}_\ell^{(1)*} \circ \widetilde{\mathbf{a}}_\ell^{(2)*} \circ \cdots \circ \widetilde{\mathbf{a}}_\ell^{(R)*}.$$

We define a tensor $\widetilde{\mathcal{U}}_s \in \mathbb{C}^{|\mathbb{S}_1|\times\cdots\times|\mathbb{S}_R|\times D}$ that collects the orthonormal bases that span $\left\{\widetilde{\mathbf{a}}_\ell^{(1)} \circ \widetilde{\mathbf{a}}_\ell^{(2)} \circ \cdots \circ \widetilde{\mathbf{a}}_\ell^{(R)}\right\}$ for $\ell = 1, \ldots, D$. The noise subspace is characterized by another tensor $\widetilde{\mathcal{U}}_n$, representing the associated orthonormal bases. The tensor MUSIC spectrum via CP is defined as

$$P_{Sparse,CP}\left(\bar{\boldsymbol{\mu}}\right) = \left\| \left\langle \widetilde{\mathcal{U}}_n, \boldsymbol{\mathcal{V}}_{\mathbb{U}^+}\left(\bar{\boldsymbol{\mu}}\right) \right\rangle_{\{1,2,\ldots,R\}} \right\|_F^{-2}.$$

Note that this alternative tensor MUSIC spectrum is based on another notion of signal/noise subspace. In HOSVD, by contrast, the noise subspace was defined as $\operatorname{span}\left\{[\widetilde{\mathbf{U}}_{1,n}]_\ell \circ [\widetilde{\mathbf{U}}_{2,n}]_\ell \circ \cdots \circ [\widetilde{\mathbf{U}}_{R,n}]_\ell\right\}$, where $[\widetilde{\mathbf{U}}_{r,n}]_\ell$ denotes the ℓ^{th} orthonormal column of $\widetilde{\mathbf{U}}_{r,n}$, for all possible ℓ. We proved in Theorem 1 that the resulting tensor MUSIC spectrum is equivalent to a separable MUSIC spectrum.

On the other hand, in the CP case, the signal subspace becomes $\operatorname{span}\left\{\widetilde{\mathbf{a}}_\ell^{(1)} \circ \widetilde{\mathbf{a}}_\ell^{(2)} \circ \cdots \circ \widetilde{\mathbf{a}}_\ell^{(R)}\right\}$, where $\ell = 1, \ldots, D$, and the noise subspace is the orthogonal complement of the signal subspace. It can be seen that in this case, the noise subspace does not have a trivial separable basis (even though the signal subspace has). Hence, with CP, it is possible to overcome the separable MUSIC spectrum problem in Theorem 1. It will be shown in Section IV that the tensor MUSIC spectrum via CP produces a finer resolution spectrum and avoids those false peaks, compared to other tensor MUSIC spectra, such as in (6) or Definition 1.

IV. Numerical Results

In this section, we consider a coprime array with parameters $M = 3$ and $N = 5$. The sensor locations are $\mathbb{S}_r = \{0, 3, 6, 9, 12, 5, 10, 15, 20, 25\}$ for $r = 1, \ldots, R$. The number of sensors in each dimension $|\mathbb{S}_r| = 10$. $\mathbb{U}_r$ are composed of consecutive integers from -17 to 17. The SNR is 0dB, sources have equal power, and the number of snapshots is 1000. The number of sources is chosen to be $D = 5$ and known to the MUSIC algorithms. Parameters $\bar{\boldsymbol{\mu}}$ are randomly drawn from their parameter space. The MATLAB Tensor Toolbox [19] is used.

The first example, as shown in the first row of Fig. 1, considers the joint angle-Doppler estimation (JADE), which is our $R = 2$ case. Here we avoid implementing the spatial smoothing step and calculate tensor MUSIC spectra alternatively as in [15]. As a comparison, we evaluate $P_{ULA,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ (tensor MUSIC spectrum in [10]), $P_{Sparse,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ (tensor MUSIC spectrum for sparse arrays and HOSVD in Definition 1), and $P_{Sparse,CP}\left(\bar{\boldsymbol{\mu}}\right)$ (tensor MUSIC spectrum for sparse arrays and CP in Definition 2). It can be seen from the results that $P_{ULA,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ and $P_{Sparse,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ come from products of two one-dimension MUSIC spectra, as Theorem 1 states. In addition, $P_{Sparse,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ produces a cleaner spectrum than $P_{ULA,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ since the lines look sharper in $P_{Sparse,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$. However, it is not obvious from the pattern where these D sources are. By exploiting the CP decomposition on $\widetilde{\mathcal{R}}$, we obtain a non-separable tensor MUSIC spectrum $P_{Sparse,CP}\left(\bar{\boldsymbol{\mu}}\right)$ on the third column of Fig. 1. $D = 5$ sources can be seen without cross-term issues.

In the second example, we set $R = 3$, which corresponds to joint estimation of azimuth, elevation, and Doppler information across a two-dimensional separable coprime array with coprime samplers at the output of sensors. In principle, the tensor MUSIC spectrum becomes a three-dimensional profile. To visualize such spectrum, the surfaces that are above a threshold in the tensor MUSIC spectrum are shown in the second row of Fig. 1. This threshold is set to be one percent of the maximal value in that spectrum. These observations are still consistent with the previous example. $P_{ULA,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ and $P_{Sparse,HOSVD}\left(\bar{\boldsymbol{\mu}}\right)$ exhibit cross terms and undesired lines while $P_{Sparse,CP}\left(\bar{\boldsymbol{\mu}}\right)$ correctly results in five distinguishable regions that disclose the actual source characteristics.

V. Concluding Remarks

In this paper, we first formulated a tensor $\widetilde{\mathcal{R}}$ that is equivalent to implementing spatial smoothing in the coarray domain. Tensor decomposition on $\widetilde{\mathcal{R}}$ allowed us to define a tensor MUSIC spectrum. Using HOSVD, it was proved that the resultant tensor MUSIC spectrum is actually a separable spectrum that comes with unwanted cross-terms. On the other hand, tensor MUSIC spectrum via CP produces a much better result than that of HOSVD.

Future research will be directed toward the theoretical analysis of these tensor approaches, such as the maximal number of detectable sources, and the Cramér-Rao lower bound for sparse arrays. Furthermore, the HOSVD and the joint Schur decomposition in the context of tensor MUSIC can also be a research topic in the future.

Appendix: Tensor Notations

- *The outer product* is $\boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{B}} \in \mathbb{C}^{I_1\times\ldots I_N\times J_1\times\ldots J_M}$ such that $\left[\boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{B}}\right]_{i_1,\ldots,i_N,j_1,\ldots,j_M} = \left[\boldsymbol{\mathcal{A}}\right]_{i_1,\ldots,i_N}\left[\boldsymbol{\mathcal{B}}\right]_{j_1,\ldots,j_M}$.
- *The inner product* ($\langle\cdot,\cdot\rangle$) between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$ is

$$\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{C}} \rangle = \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} \left[\boldsymbol{\mathcal{A}}\right]^*_{i_1,\ldots,i_N}\left[\boldsymbol{\mathcal{C}}\right]_{i_1,\ldots,i_N}.$$

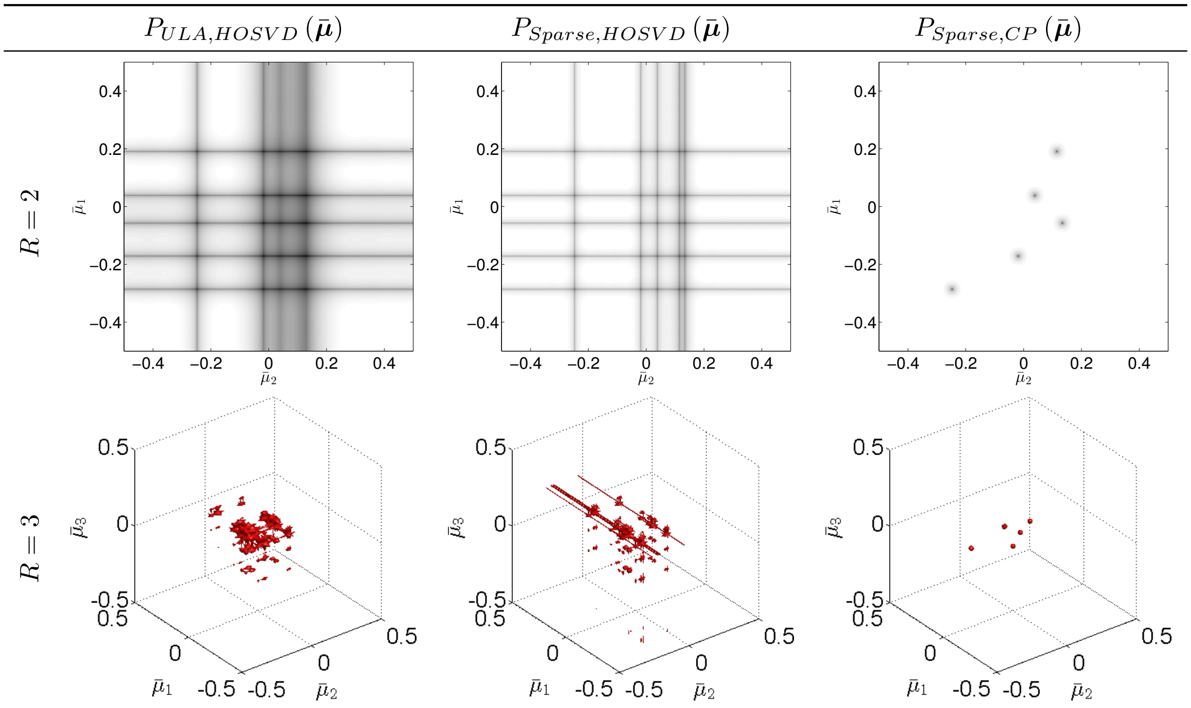

Fig. 1. Tensor MUSIC spectrum for $D = 5$ sources distributed at random in a R-dimensional tensor model. The maximal value of each MUSIC spectrum is normalized to be 1. In the $R = 2$ case, darker regions denote larger values in the MUSIC spectrum. When $R = 3$, the red portions indicate locations where MUSIC spectrum is above 0.01.

- *The contraction product* $\langle \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}} \rangle_{\{1,\ldots,N\}}$ satisfies

$$\left[\langle \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}} \rangle_{\{1,\ldots,N\}}\right]_{j_1,\ldots,j_M,k_1,\ldots,k_O} = \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} [\boldsymbol{\mathcal{D}}]^*_{i_1,\ldots,i_N,j_1,\ldots,j_M} [\boldsymbol{\mathcal{E}}]_{i_1,\ldots,i_N,k_1,\ldots,k_O},$$

  where $1 \le i_p \le I_p$, $1 \le j_q \le J_q$, $1 \le k_r \le K_r$, $p = 1, \ldots, N$, $q = 1, \ldots, M$, and $r = 1, \ldots, O$.
- The *Frobenius norm* $\|\cdot\|_F$ of $\boldsymbol{\mathcal{A}}$ is $\|\boldsymbol{\mathcal{A}}\|_F = \sqrt{\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}} \rangle}$.
- *The* n*-mode product* $(\times_n)$ takes a tensor $\boldsymbol{\mathcal{A}}$ and a matrix $\mathbf{U} \in \mathbb{C}^{J_n \times I_n}$ such that

$$[\boldsymbol{\mathcal{A}} \times_n \mathbf{U}]_{i_1,\ldots,i_{n-1},j_n,i_{n+1},\ldots,i_N} = \sum_{i_n=1}^{I_n} [\boldsymbol{\mathcal{A}}]_{i_1,\ldots,i_N} [\mathbf{U}]_{j_n,i_n},$$

  where $1 \le i_\ell \le I_\ell$, $1 \le j_n \le J_n$, and $\ell = 1, \ldots, N$.

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