Robustness of Coarrays of Sparse Arrays to Sensor Failures

Chun-Lin Liu[1] and P. P. Vaidyanathan[2]

Dept. of Electrical Engineering, MC 136-93
California Institute of Technology,
Pasadena, CA 91125, USA
cl.liu@caltech.edu[1], ppvnath@systems.caltech.edu[2]

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Outline

1 Introduction

2 Review of Sparse Arrays

3 The Essentialness Property and the Fragility

4 Numerical Examples

5 Concluding Remarks

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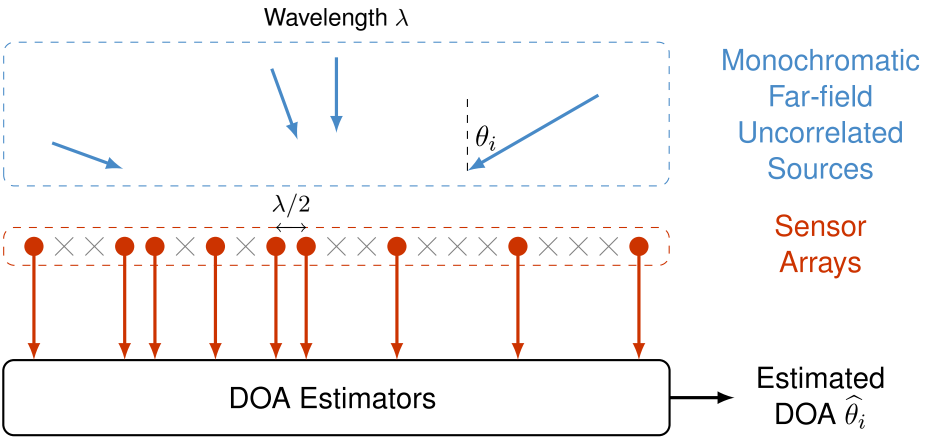
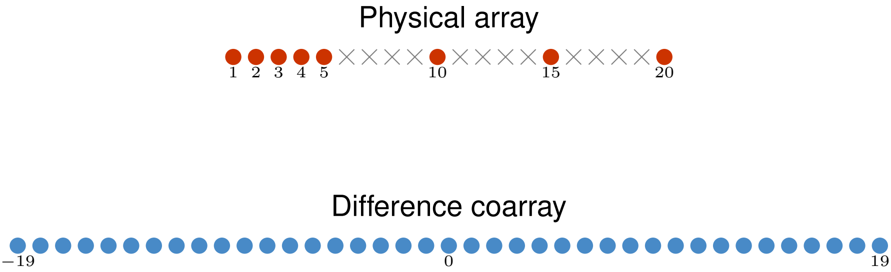
Direction-Of-Arrival (DOA) Estimation



[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

Physical Array and Difference Coarray



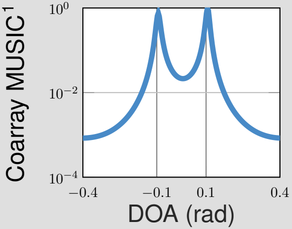
[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

Sensor Failures

Array #1

5 elements

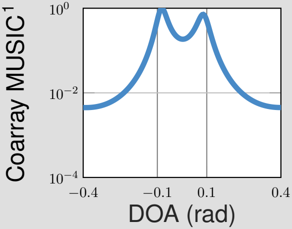
$\mathrm{RMSE} = 0.00617$



Array #2 (2 fails)

4 elements

$\mathrm{RMSE} = 0.014367$



Array #3 (1 fails)

4 elements

Coarray MUSIC is not applicable here!

[1] Liu and Vaidyanathan, *IEEE Signal Process. Letters*, 2015.

[2] 100 snapshots, 0dB SNR, $D = 2$ sources, $\theta_1 = -0.1$, $\theta_2 = 0.1$, equal-power, uncorrelated sources.

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ULA and Sparse Arrays

ULA (not sparse)

- Identify at most $N_{\text{sensors}} - 1$ uncorrelated sources.[1]

  (N_{sensors} is the number of sensors)
- Can only find fewer sources than sensors.

Linear sparse arrays

1. Minimum redundancy arrays[2]
2. Nested arrays[3]
3. Coprime arrays[4]
4. Super nested arrays[5]

- Identify $O(N_{\text{sensors}}^2)$ uncorrelated sources.
- More sources than sensors!

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968.

[3] Pal and Vaidyanathan, *IEEE Trans. Signal Process.*, 2010.

[4] Vaidyanathan and Pal, *IEEE Trans. Signal Process.*, 2011.

[5] Liu and Vaidyanathan, *IEEE Trans. Signal Process.*, 2016.

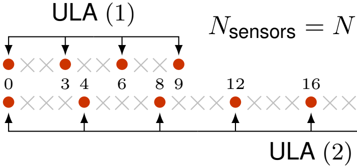
Coprime Arrays

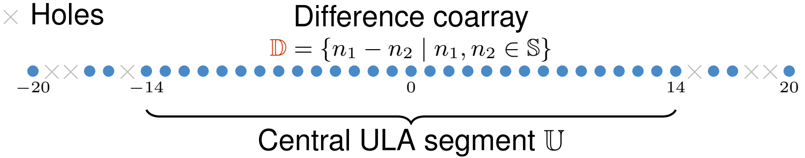
The coprime array with $(M, N) = 1$ is the union of

1. an N-element ULA with spacing M and
2. a $2M$-element ULA with spacing N.

$N_{\text{sensors}} = N + 2M - 1$

Physical array $\mathbb{S}$ $(M = 3, N = 4)$:





[1] Vaidyanathan and Pal, *IEEE Trans. Signal Process.*, 2011.

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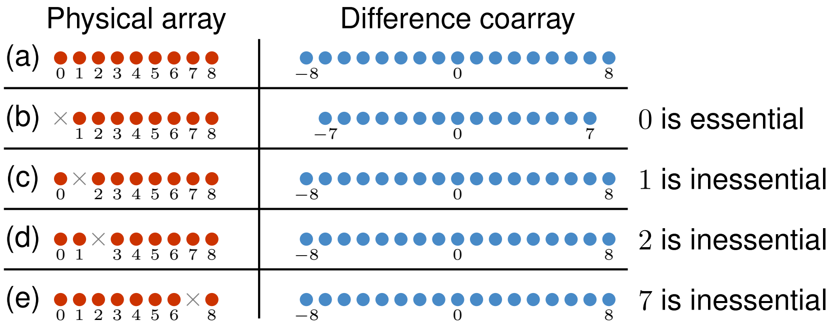
The Essentialness Property

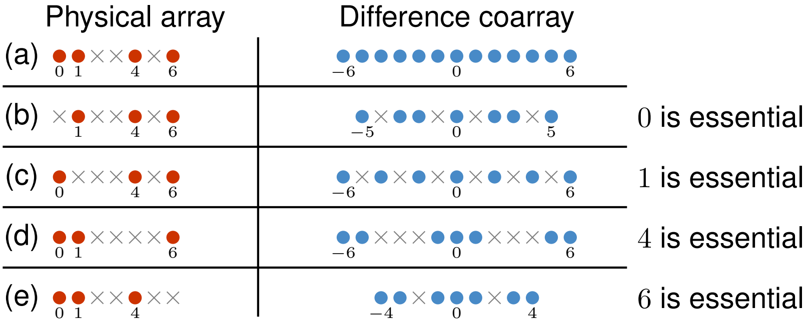
The sensor $n \in \mathbb{S}$ is essential with respect to $\mathbb{S}$ if $\overline{\mathbb{D}} \neq \mathbb{D}$.



[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018; $\mathbb{D}$ is the difference coarray of $\mathbb{S}$ and $\overline{\mathbb{D}}$ is the difference coarray of $\mathbb{S}\backslash\{n\}$.

Maximally Economic Sparse Arrays

An array $\mathbb{S}$ is maximally economic
if all the sensors in $\mathbb{S}$ are essential



Array (a) is maximally economic

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

Maximally Economic Sparse Arrays

- Array geometries that are maximally economic:

Minimum redundancy array

Minimum hole array

Nested array

Cantor array

- Array geometries that are not maximally economic:

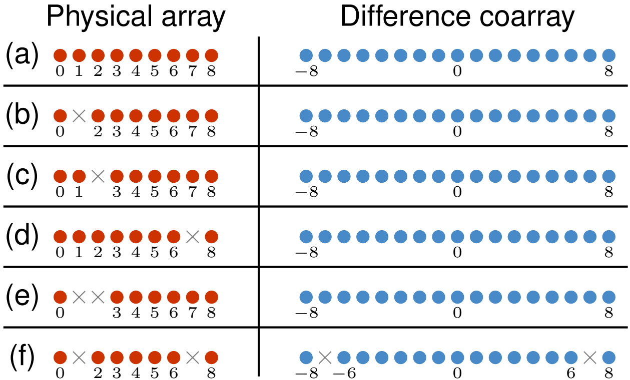
Uniform linear array

Coprime array

[1] Liu and Vaidyanathan, *IEEE CAMSAP*, 2017; Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

The k-Essentialness Property

The set $\mathbb{A}$ is *k-essential* with respect to $\mathbb{S}$
if $\mathbb{A} \subseteq \mathbb{S}$, $|\mathbb{A}| = k$, and $\overline{\mathbb{D}} \neq \mathbb{D}$.



(b) $\{1\}$ is not 1-essential

(c) $\{2\}$ is not 1-essential

(d) $\{7\}$ is not 1-essential

(e) $\{1, 2\}$ is not 2-essential

(f) $\{1, 7\}$ is 2-essential

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018; $\mathbb{D}$ is the difference coarray of $\mathbb{S}$ and $\overline{\mathbb{D}}$ is the difference coarray of $\mathbb{S}\backslash\mathbb{A}$.

The k-Essential Family $\mathcal{E}_k$: Definition

The k-essential family: $\mathcal{E}_k \triangleq \{\mathbb{A} : \mathbb{A} \text{ is } k\text{-essential}\}$

All subarrays of size k: $\mathcal{S}_k \triangleq \{\mathbb{A} \subseteq \mathbb{S} : |\mathbb{A}| = k\}$

Example: $\mathbb{S} = \{0, 1, 2, 3\}$

$$\mathcal{E}_1 = \{\{0\}, \{3\}\},$$

$$\mathcal{E}_2 = \{\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}\} = \mathcal{S}_2,$$

$$\mathcal{E}_3 = \{\{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}\} = \mathcal{S}_3,$$

$$\mathcal{E}_4 = \{\{0,1,2,3\}\} = \mathcal{S}_4.$$

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

The k-Essential Family $\mathcal{E}_k$: Proposition

The k-essential family: $\mathcal{E}_k \triangleq \{\mathbb{A} : \mathbb{A} \text{ is } k\text{-essential}\}$

All subarrays of size k: $\mathcal{S}_k \triangleq \{\mathbb{A} \subseteq \mathbb{S} : |\mathbb{A}| = k\}$

Proposition

1. The size of the k-essential family satisfies

$$(|\mathbb{S}| - k)|\mathcal{E}_k| \leq (k+1)|\mathcal{E}_{k+1}|,$$

   for all $1 \leq k \leq |\mathbb{S}| - 1$.
   The equality holds if and only if $\mathcal{E}_k = \mathcal{S}_k$.

2. $\mathcal{E}_k = \mathcal{S}_k$ for all $|\mathbb{S}| - |\mathcal{E}_1| + 1 \leq k \leq |\mathbb{S}|$.

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

The k-Essential Family $\mathcal{E}_k$: Proposition

Proposition 1. Let $\mathcal{E}_k$ be the k-essential family with respect to a nonempty integer set $\mathbb{S}$. Then the following properties hold true:

1. $(|\mathbb{S}| - k)|\mathcal{E}_k| \leq (k+1)|\mathcal{E}_{k+1}|$ for all $1 \leq k \leq |\mathbb{S}| - 1$. The equality holds if and only if $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$.
2. $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$ for all $|\mathbb{S}| - |\mathcal{E}_1| + 1 \leq k \leq |\mathbb{S}|$.
3. If $\mathbb{S}$ is maximally economic, then $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$ for all $1 \leq k \leq |\mathbb{S}|$.
4. Let $M_p = |\{m \in \mathbb{D} : w(m) = p\}|$ be the number of elements in the difference coarray such that the associated weight function is p. If $|\mathbb{S}| \geq 2$, then

$$\left\lceil \frac{1 + \sqrt{1 + 4M_1}}{2} \right\rceil \leq |\mathcal{E}_1| \leq \min\left\{ M_1 + \left\lfloor \frac{M_2}{2} \right\rfloor, |\mathbb{S}| \right\}, \tag{5}$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are the ceiling function and the floor function, respectively.

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

The k-Fragility (or Fragility): Definition

The k-fragility $F_k \triangleq \frac{|\mathcal{E}_k|}{|\mathcal{S}_k|} = \frac{\text{\# of } k\text{-essential subarrays}}{\text{\# of all subarrays of size } k}$

$$0 \leq F_k \leq 1 \text{ for all } 1 \leq k \leq |\mathbb{S}|$$

More robust
Less fragile

F_k

Less robust
More fragile

0 1

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

The k-Fragility: Proposition

The k-fragility $F_k \triangleq \frac{|\mathcal{E}_k|}{|\mathcal{S}_k|} = \frac{\text{\# of } k\text{-essential subarrays}}{\text{\# of all subarrays of size } k}$

Proposition

1. $F_k \leq F_{k+1}$ for all $1 \leq k \leq |\mathbb{S}| - 1$.

   The equality holds if and only if $F_k = 1$.

2. $F_k = 1$ for all $|\mathbb{S}| - |\mathcal{E}_1| + 1 \leq k \leq |\mathbb{S}|$.

3. If $\mathbb{S}$ is maximally economic, then $F_k = 1$ for all $1 \leq k \leq |\mathbb{S}|$.

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

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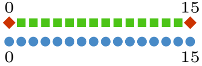
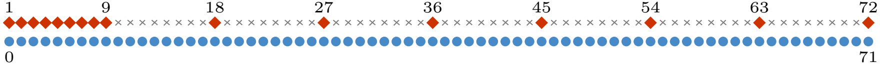
Array Geometry

(a): Uniform linear array with 16 elements $|\mathcal{E}_1| = 2$

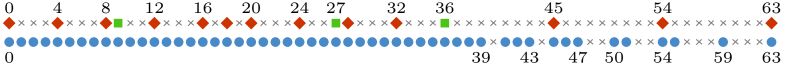


◆ Essential ■ Inessential ● Coarray

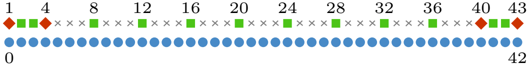
(b): Nested array with 16 elements [Pal and Vaidyanathan, 2010] $|\mathcal{E}_1| = 16$



(c): Coprime array with 16 elements [Vaidyanathan and Pal, 2011] $|\mathcal{E}_1| = 13$

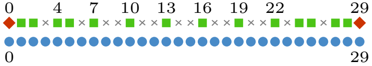


(d): Concatenated nested array with 16 elements $|\mathcal{E}_1| = 4$



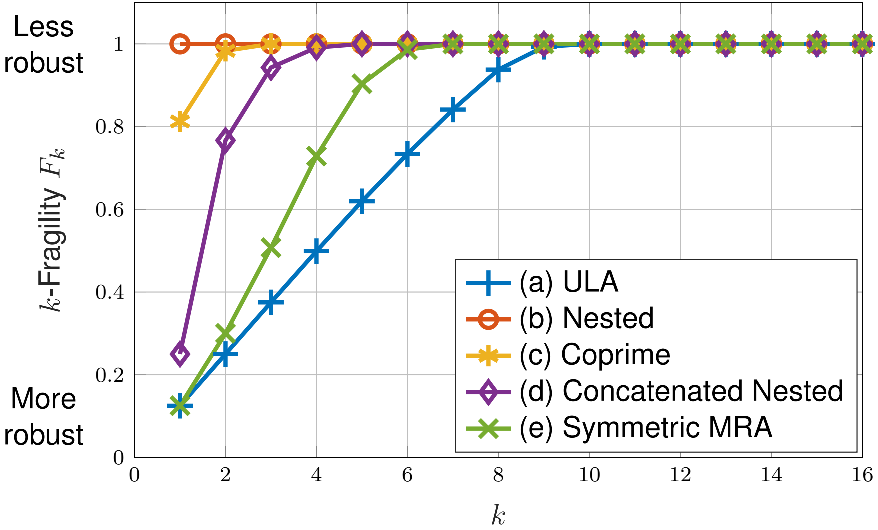
[Rajamäki and Koivunen, 2017]

(e): Symmetric MRA with 16 elements $|\mathcal{E}_1| = 2$



[Liu and Vaidyanathan, 2017]

The k-Fragility F_k



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Concluding Remarks

- Robustness of difference coarrays to sensor failures
  - The k-essentialness property
  - The k-essential family
  - The k-fragility
  - Comparison of sparse arrays
- Future work
  - Probability that the difference coarray changes
  - Robustness of the central ULA segment

Thank you!