ONE-BIT SPARSE ARRAY DOA ESTIMATION

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ABSTRACT

One-bit quantization has become an important topic in massive MIMO systems, as it offers low cost and low complexity in the implementation. Techniques to achieve high performance in spite of the coarse quantizers have recently been advanced. In the context of array processing and direction-of-arrival (DOA) estimation also, one bit quantizers have been studied in the past, although not as extensively. This paper shows that sparse arrays such as nested and coprime arrays are more robust to the deleterious effects of one-bit quantization, compared to uniform linear arrays (ULAs); in fact, sparse arrays with one-bit quantizers are often found to be as good as ULAs with unquantized data. Nested and coprime arrays without quanitzers are known to be able to resolve more DOAs than the number of sensors, when sources are uncorrelated. It will be demonstrated that this continues to be true even with one-bit quantization.

***Index Terms*—** One-bit quantization, sparse arrays, nested arrays, coprime arrays, DOA estimation.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation, which calculates the source directions from sensor measurements, arises in many important topics such as radar, beamforming, imaging, and communications [1–5]. In these applications, the sensor measurements over multiple snapshots are collected first. Then one calculates the sample covariance matrix, from which the DOA estimates are obtained. However, it is expensive to represent sensor measurements using many bits, as it requires high speed and high resolution analog-to-digital converters (ADC), adders, and multipliers. The power consumption of such systems is also an important issue [6].

Recently, one-bit quantization, which records the sign of real and imaginary parts of measurements, has been gaining momentum in massive MIMO systems [7–10]. One-bit quantizers offer significantly low cost and low complexity, while still maintaining good performance in massive MIMO systems [10, 11]. Furthermore, in circuit design, it is much simpler to design, calibrate, and implement the one-bit ADC, than the high resolution ones [6]. In array processing also, the one-bit quantizers have been used in DOA estimation [7, 12, 13]. In particular, it was demonstrated in [13] that, for uniform linear arrays (ULA) and one Gaussian source, the performance loss introduced by one-bit quantization is moderate.

The DOA estimation performance depends highly on the array configurations. It is well-known that ULAs with N sensors can resolve up to $N-1$ noncoherent sources [4]. Sparse arrays such as nested and coprime arrays can identify $O(N^2)$ uncorrelated sources, and also enjoy a smaller Cramér-Rao bound for DOA estimation error, than ULAs [14–16], since the difference coarrays for sparse arrays contain $O(N^2)$ consecutive integers. Some popular sparse arrays include minimum redundancy arrays (MRA) [17], nested arrays [14], coprime arrays [18], and some generalizations [19–21]. Unlike MRAs, nested arrays and coprime arrays possess closed-form sensor locations. It was demonstrated amply that spatial smoothing MUSIC (SS MUSIC) can estimate more DOAs than sensors using sparse arrays [14, 15].

In this paper, we will propose a new DOA estimator using one-bit measurements on sparse arrays. While the covariance matrix of the unquantized data cannot be recovered from the quantized data, the so-called *normalized covariance* of the unquantized data can still be estimated, by using some extensions [22] of the well-known arcsine law [23] in statistics. We will show that by performing SS MUSIC based on this normalized covariance, the DOAs can still be estimated satisfactorily. The performance of the proposed estimator will be assessed through numerical examples. For fixed number of sensors, the proposed estimator using one-bit data on sparse arrays can even outperform the ULA with *unquantized data*, for a wide range of parameters. Thus, the degradation due to one-bit quantization can, to a significant extent, be countered using sparse arrays. We will also demonstrate that sparse arrays with one-bit quantizers can still resolve more DOAs than the number of sensors, as they do in the unquantized case [14–16].

The outline of this paper is as follows: Section 2 and 3 review sparse arrays and one-bit quantization, respectively. An one-bit sparse array DOA estimator based on SS MUSIC, is proposed in Section 4. Section 5 gives several examples for the proposed estimator while Section 6 concludes this paper.

2. DOA ESTIMATION USING SPARSE ARRAYS

Assume that D monochromatic, far-field, and uncorrelated sources illuminate an one-dimensional sensor array, where the sensors are located at position $n\lambda/2$. Here λ is the wavelength and n belongs to an integer set $\mathbb{S}$. The sensor measurements can be modeled as

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}\left(\bar{\theta}_i\right) + \mathbf{n}_{\mathbb{S}} \quad \in \mathbb{C}^{|\mathbb{S}|}, \tag{1}$$

where $A_i \in \mathbb{C}$, $\theta_i \in [-\pi/2, \pi/2]$, and $\bar{\theta}_i = (\sin\theta_i)/2 \in [-1/2, 1/2]$ denote the complex amplitude, the DOA, and the normalized DOA of the ith source, respectively. The steering vector of the ith source is defined as $\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) = [e^{j2\pi\bar{\theta}_i n}]_{n\in\mathbb{S}}$ and $\mathbf{n}_{\mathbb{S}}$ is the additive noise term. The cardinality of $\mathbb{S}$ is denoted by $|\mathbb{S}|$. Let $\mathbf{s} = [A_1, \ldots, A_D, \mathbf{n}_{\mathbb{S}}^T]^T$. It is assumed that $\mathbf{s}$ is a circularly-symmetric complex Gaussian random vector satisfying $\mathbb{E}[\mathbf{s}] = \mathbf{0}$ and

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$\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \text{diag}(p_1, \ldots, p_D, p_n, \ldots, p_n)$. Here $\text{diag}(a_1, \ldots, a_N)$ is the diagonal matrix with diagonals $a_1, \ldots, a_N$. The ith source power and the noise power are denoted by p_i and p_n, respectively.

The covariance matrix of $\mathbf{x}_{\mathbb{S}}$ can be expressed as

$$\mathbf{R}_{\mathbf{x}_{\mathbb{S}}} = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + p_n\mathbf{I} \quad \in \mathbb{C}^{|\mathbb{S}|\times|\mathbb{S}|}. \tag{2}$$

Vectorizing and combining duplicate entries in (2) give the correlation vector $\mathbf{x}_{\mathbb{D}}$ on the difference coarray:

$$\mathbf{x}_{\mathbb{D}} = \mathbf{J}^{\dagger}\text{vec}(\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}) = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{D}}(\bar{\theta}_i) + p_n\mathbf{e}_0 \quad \in \mathbb{C}^{|\mathbb{D}|}, \tag{3}$$

where the difference coarray $\mathbb{D}$ is defined as

Definition 1 (Difference coarray)**.** *For an array specified by an integer set $\mathbb{S}$, its difference coarray $\mathbb{D}$ is defined as $\mathbb{D} = \{n_1 - n_2 \mid \forall n_1, n_2 \in \mathbb{S}\}$.*

The column vector $\mathbf{e}_0$ in (3) satisfies $\langle\mathbf{e}_0\rangle_m = \delta_{m,0}$, where $m \in \mathbb{D}$ and $\delta_{i,j}$ is the Kronecker delta. Here $\langle\cdot\rangle_m$ denotes the signal value on the support $m \in \mathbb{D}$ [24]. For instance, if $\mathbb{D} = \{-3, 0, 3\}$ and $\mathbf{x}_{\mathbb{D}} = [2 + j, 3, 2 - j]^T$, then $\langle\mathbf{x}_{\mathbb{D}}\rangle_{-3} = 2 + j$, $\langle\mathbf{x}_{\mathbb{D}}\rangle_0 = 3$, and $\langle\mathbf{x}_{\mathbb{D}}\rangle_3 = 2 - j$. The term $\mathbf{J}^{\dagger}$ in (3) denotes the Moore-Penrose pseudoinverse of $\mathbf{J}$, where $\mathbf{J}$ is defined as follows [16]

Definition 2 (The matrix $\mathbf{J}$)**.** *The binary matrix $\mathbf{J}$ has size $|\mathbb{S}|^2$-by-$|\mathbb{D}|$. The columns of $\mathbf{J}$ satisfy $\langle\mathbf{J}\rangle_{:,m} = \text{vec}(\mathbf{I}(m))$ for $m \in \mathbb{D}$, where $\mathbf{I}(m) \in \{0, 1\}^{|\mathbb{S}|\times|\mathbb{S}|}$ is given by*

$$\langle\mathbf{I}(m)\rangle_{n_1,n_2} = \begin{cases} 1, & \text{if } n_1 - n_2 = m, \\ 0, & \text{otherwise.} \end{cases} \quad \forall n_1, n_2 \in \mathbb{S}.$$

Furthermore, the weight function $w(m)$ is defined as the number of nonzero entries in $\mathbf{I}(m)$. It can be shown that $\mathbf{J}^H\mathbf{J} = \text{diag}(w(m))_{m\in\mathbb{D}}$ [16]. The model for $\mathbf{x}_{\mathbb{D}}$, as in (3), can be regarded as D coherent sources on $\mathbb{D}$. This property admits DOA estimators on $\mathbf{x}_{\mathbb{D}}$, such as SS MUSIC [14, 15, 24].

The most important merit of DOA estimators on $\mathbf{x}_{\mathbb{D}}$ is that, $O(N^2)$ uncorrelated sources can be resolved using only $O(N)$ physical sensors. This is because the size of $\mathbf{x}_{\mathbb{D}}$ is $|\mathbb{D}| = O(N^2)$ for some sparse arrays. In particular, let the central ULA segment of $\mathbb{D}$ be the set $\mathbb{U}$. Then, up to $(|\mathbb{U}| - 1)/2$ uncorrelated sources can be identified by SS MUSIC [14, 15]. For instance, the coprime array in Fig. 1(c) has the difference coarray $\mathbb{D} = \{0, \pm1, \ldots, \pm17, \pm19, \pm20, \pm22, \pm25\}$, and the set $\mathbb{U} = \{0, \pm1, \ldots, \pm17\}$. Hence SS MUSIC can identify $(|\mathbb{U}|-1)/2 = 17$ uncorrelated sources.

Nested arrays [14] and coprime arrays [15,18] are two classes of sparse arrays that satisfy $|\mathbb{U}| = O(N^2)$ with $O(N)$ physical sensors. A nested array with $N_1 + N_2$ sensors, as depicted in Fig. 1(b), has the following sensor locations:

$$\mathbb{S}_{\text{nested}} = \{1, \ldots, N_1, (N_1 + 1), \ldots N_2(N_1 + 1)\}, \tag{4}$$

where N_1 and N_2 are positive integers. The difference coarrays for nested arrays are exactly ULAs, namely, $\mathbb{D}_{\text{nested}} = \mathbb{U}_{\text{nested}} = \{0, \pm1, \ldots, \pm(N_2(N_1 + 1) - 1)\}$. Fig. 1(c) illustrates a coprime array with $N + 2M - 1$ sensors, where M and N are a coprime pair of positive integers. The sensor locations for coprime arrays are

$$\mathbb{S}_{\text{coprime}} = \{0, M, \ldots, (N-1)M, N, \ldots, (2M-1)N\}. \tag{5}$$

The difference coarrays for coprime arrays have a long ULA segment $\mathbb{U}_{\text{coprime}} = \{0, \pm1, \ldots, \pm(MN + M - 1)\}$, and some missing elements (holes) outside $\mathbb{U}_{\text{coprime}}$ [15, 19].

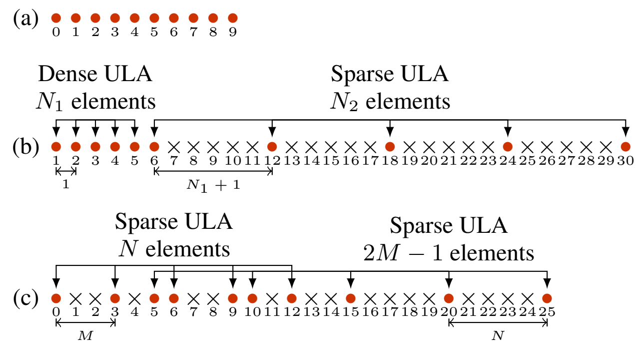

Fig. 1. The array configurations for (a) ULA with 10 sensors, (b) a nested array with $N_1 = N_2 = 5$, and (c) a coprime array with $M = 3$ and $N = 5$. Here bullets denote sensors while crosses represent empty space.

3. ONE-BIT QUANTIZATION

In this section, we will review some statistical properties of one-bit quantization, such as the arcsine law for real scalars [23, 25], the Bussgang theorem [26], and the arcsine law for complex vectors [7, 22, 27], which is valuable for one-bit DOA estimators.

Historically speaking, the autocorrelation function of one-bit data was studied by Van Vleck and Middleton [23, 25]. They considered a continuous-time, *real*, *scalar*, and stationary Gaussian process $X(t)$ and the output process $Y(t) = f(X(t))$, where $f(x)$ is a memoryless amplitude-distortion function. If $f(x)$ is the sign function (one-bit quantizer), then the autocorrelation function of $Y(t)$, denoted by $R_Y(\tau)$, is described by the ***arcsine law***:

$$R_Y(\tau) \triangleq \mathbb{E}[Y(t + \tau)Y(t)] = \frac{2}{\pi}\sin^{-1}\bar{R}_X(\tau), \tag{6}$$

where $\bar{R}_X(\tau) = R_X(\tau)/R_X(0)$ is the normalized autocorrelation function of $X(t)$. Hence, by observing the one-bit data in $Y(t)$, we can recover the normalized autocorrelation of the unquantized data. This elegant result has been applied to radio astronomy [28], biomedical engineering [29], and communications [30].

Furthermore, under the same assumption, ***the Bussgang theorem*** [26] states that, the cross-correlation between $X(t)$ and $Y(t)$ is proportional to the autocorrelation of $X(t)$. That is, $R_{XY}(\tau) \triangleq \mathbb{E}[X(t+\tau)Y(t)] = CR_X(\tau)$, where the factor C depends purely on the characteristics of $f(x)$ and the power of $X(t)$. The significance of the Bussgang theorem is that, the output process $Y(t)$, which is typically a nonlinear function of $X(t)$, is equivalent to a linear function of $X(t)$, as far as second order statistics is concerned. This idea has found applications in communications [31], and even in neuroscience [32].

Note that (6) applies if the process $X(t)$ is real and scalar. In the following development, we will consider the complex vector case [7, 22, 27]. For the complex vector $\mathbf{x}_{\mathbb{S}}$ in (1), the one-bit quantized measurement vector $\mathbf{y}_{\mathbb{S}}$ is defined as

$$\mathbf{y}_{\mathbb{S}} = \frac{1}{\sqrt{2}}\text{sgne}(\mathbf{x}_{\mathbb{S}}) \quad \in \mathbb{C}^{|\mathbb{S}|}, \tag{7}$$

where the qth entry of $\text{sgne}(\mathbf{x}_{\mathbb{S}})$ is given by $[\text{sgne}(\mathbf{x}_{\mathbb{S}})]_q = \text{sgn}(\text{Re}([\mathbf{x}_{\mathbb{S}}]_q)) + j\text{sgn}(\text{Im}([\mathbf{x}_{\mathbb{S}}]_q))$. Here the sign function $\text{sgn}(x)$ is 1 if x is nonnegative and -1 otherwise. $\text{Re}(z)$ and $\text{Im}(z)$ denote the real and the imaginary parts of z, respectively. The factor $1/\sqrt{2}$ normalizes the power of $\mathbf{y}_{\mathbb{S}}$. So the entries of $\mathbf{y}_{\mathbb{S}}$ take four discrete values $(\pm1 \pm j)/\sqrt{2}$.

Next we will review the arcsine law for complex Gaussian vectors. First, the normalized covariance matrix of $\mathbf{x}_{\mathbb{S}}$ is defined as

$$\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}} = \mathbf{Q}^{-1/2}\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}\mathbf{Q}^{-1/2}, \tag{8}$$

where $\mathbf{Q}$ is a diagonal matrix satisfying $[\mathbf{Q}]_{q,q} = [\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}]_{q,q}$. Unlike the scalar case, $\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$ is not necessarily a scalar multiple of $\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}$. Then, the arcsine law for $\mathbf{x}_{\mathbb{S}}$ and $\mathbf{y}_{\mathbb{S}}$ is

$$\mathbf{R}_{\mathbf{y}_{\mathbb{S}}} \triangleq \mathbb{E}[\mathbf{y}_{\mathbb{S}}\mathbf{y}_{\mathbb{S}}^H] = \frac{2}{\pi}\text{sine}^{-1}(\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}), \tag{9}$$

where the (p,q)th element of $\text{sine}^{-1}(\mathbf{A})$ is $[\text{sine}^{-1}(\mathbf{A})]_{p,q} = \sin^{-1}(\text{Re}(A_{p,q})) + j\sin^{-1}(\text{Im}(A_{p,q}))$. Here $A_{p,q}$ denotes the (p,q)th entry of $\mathbf{A}$. Note that the arcsine law for complex vectors (9) resembles that for real scalars (6), except for the definition of the arcsine function. It can also be shown from (8) that the real and imaginary parts of the entries of $\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$ are between -1 and 1.

The implication of (9) is, the normalized covariance $\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$ can be estimated from the covariance of one-bit data, based on the following expression [7, 22, 27]:

$$\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}} = \text{sine}\left(\frac{\pi}{2}\mathbf{R}_{\mathbf{y}_{\mathbb{S}}}\right), \tag{10}$$

where the (p,q)th element of $\text{sine}(\mathbf{A})$ is given by $[\text{sine}(\mathbf{A})]_{p,q} = \sin(\text{Re}(A_{p,q})) + j\sin(\text{Im}(A_{p,q}))$.

4. ONE-BIT DOA ESTIMATORS WITH SPARSE ARRAYS

According to (10), the normalized covariance $\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$, instead of the original one $\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}$, can be recovered from $\mathbf{R}_{\mathbf{y}_{\mathbb{S}}}$. However, most DOA estimators require the original covariance matrix $\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}$. Next we will show that, $\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$ is enough for the purpose of DOA estimation if sources are uncorrelated. Similar results were reported for one source and ULAs [7, 13]. Here we assume multiple uncorrelated sources and sparse arrays.

The proposed one-bit DOA estimator using sparse arrays is summarized as follows:

1. Define the normalized correlation vector $\overline{\mathbf{x}}_{\mathbb{D}}$ as

$$\overline{\mathbf{x}}_{\mathbb{D}} = \mathbf{J}^{\dagger}\text{vec}\left(\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}\right) = \mathbf{J}^{\dagger}\text{vec}\left(\text{sine}\left(\frac{\pi}{2}\mathbf{R}_{\mathbf{y}_{\mathbb{S}}}\right)\right). \tag{11}$$

2. Let the normalized correlation vector on the ULA segment of the coarray be $\overline{\mathbf{x}}_{\mathbb{U}}$. Construct a Hermitian Toeplitz matrix $\overline{\mathbf{R}}$ satisfying $\langle\overline{\mathbf{R}}\rangle_{n_1,n_2} = \langle\overline{\mathbf{x}}_{\mathbb{U}}\rangle_{n_1-n_2}$, as formulated in [24]. Here $n_1, n_2 \in \mathbb{U}^+$, which is the nonnegative part of $\mathbb{U}$.

3. Eigen-decompose the matrix $\overline{\mathbf{R}}$ and split the signal and noise subspace according to the *magnitude* of the eigenvalues of $\overline{\mathbf{R}}$. Let the orthonormal bases of the noise subspace be the columns of $\mathbf{U}_n$.

4. Calculate the MUSIC spectrum by

$$H_{\text{normalized}}(\bar{\theta}) = \frac{1}{\left\|\mathbf{U}_n^H\mathbf{v}_{\mathbb{U}^+}(\bar{\theta})\right\|_2^2},$$

and locate the peaks in $P(\bar{\theta})$. Here $\mathbf{v}_{\mathbb{U}^+}(\bar{\theta})$ are the steering vectors on the nonnegative part of the set $\mathbb{U}$. The peak locations are the estimated normalized DOAs.

A fundamental question here is, what are the differences between the SS MUSIC spectrum $H_{\text{normalized}}(\bar{\theta})$ and the SS MUSIC spectrum $H_{\text{original}}(\bar{\theta})$ derived from the original covariance matrix $\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}$? It will be shown that $H_{\text{normalized}}(\bar{\theta}) = H_{\text{original}}(\bar{\theta})$, implying that, with sufficient snapshots, SS MUSIC with the quantized data $\mathbf{y}_{\mathbb{S}}$ makes no difference from SS MUSIC with the unquantized data $\mathbf{x}_{\mathbb{S}}$. This claim is due to the following lemma:

Lemma 1. *Assume the sources are uncorrelated. Then, with* $\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}$ *and* $\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$ *defined as in Eqs.* (2) *and* (8)*, we have* $\mathbf{R}_{\mathbf{x}_{\mathbb{S}}} = P\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$*, where* $P = \sum_{i=1}^{D} p_i + p_n > 0$ *is the total power.*

Proof. According to (8), the diagonal entry of $\mathbf{Q}$ associated with the sensor location n_1, is given by

$$\begin{aligned}\langle\mathbf{Q}\rangle_{n_1,n_1} &= \left\langle\sum_{i=1}^{D} p_i\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + p_n\mathbf{I}\right\rangle_{n_1,n_1} \\ &= \sum_{i=1}^{D} p_i e^{j2\pi\bar{\theta}_i n_1}e^{-j2\pi\bar{\theta}_i n_1} + p_n = P,\end{aligned}$$

which implies $\mathbf{Q} = P\mathbf{I}$. Using (8) proves this lemma. □

In other words, for uncorrelated sources, the normalized covariance matrix of $\mathbf{x}_{\mathbb{S}}$ is a positive scaled version of the covariance matrix of $\mathbf{x}_{\mathbb{S}}$. Combining Lemma 1, (3), and (11) gives $\mathbf{x}_{\mathbb{D}} = P\,\overline{\mathbf{x}}_{\mathbb{D}}$. Let the Hermitian Toeplitz matrix corresponding to $\mathbf{R}_{\mathbf{x}_{\mathbb{S}}}$ be $\mathbf{R}$. We have $\mathbf{R} = P\overline{\mathbf{R}}$, so $\mathbf{R}$ and $\overline{\mathbf{R}}$ share the same noise subspace. Hence $H_{\text{normalized}}(\bar{\theta}) = H_{\text{original}}(\bar{\theta})$.

It is known that, with sufficient snapshots, $O(N^2)$ uncorrelated sources can be resolved using the original correlation vector $\mathbf{x}_{\mathbb{D}}$, if sparse arrays, like nested arrays or coprime arrays, with $O(N)$ sensors are deployed. Since $\mathbf{x}_{\mathbb{D}} = P\,\overline{\mathbf{x}}_{\mathbb{D}}$, the same argument holds true for the normalized covariance matrx $\overline{\mathbf{R}}_{\mathbf{x}_{\mathbb{S}}}$ and the normalized correlation vector $\overline{\mathbf{x}}_{\mathbb{D}}$. Namely, even with one-bit measurements, it is still possible to identify more sources than sensors using sparse arrays, if there are enough snapshots. This claim will be verified through examples in Section 5.

The above discussion assumes ideal covariance matrices. In the following developement, we will consider the finite snapshot scenario, in which the normalized correlation vector $\overline{\mathbf{x}}_{\mathbb{D}}$ is replaced with its finite snapshot version, whereas the remaining steps are still applicable. Let the one-bit measurements be $\widetilde{\mathbf{y}}_{\mathbb{S}}(k)$ for $k = 1, \ldots, K$. Due to (11), the normalized correlation vector can be estimated as

$$\widetilde{\overline{\mathbf{x}}}_{\mathbb{D}} = \mathbf{J}^{\dagger}\text{vec}\left(\text{sine}\left(\frac{\pi}{2K}\sum_{k=1}^{K}\widetilde{\mathbf{y}}_{\mathbb{S}}(k)\widetilde{\mathbf{y}}_{\mathbb{S}}^H(k)\right)\right). \tag{12}$$

Eq. (12) has several advantages in terms of hardware implementation. First of all, evaluating $\sum_{k=1}^{D}\widetilde{\mathbf{y}}_{\mathbb{S}}(k)\widetilde{\mathbf{y}}_{\mathbb{S}}^H(k)$ requires only addition, because $\widetilde{\mathbf{y}}_{\mathbb{S}}(k)$ take only four values $(\pm 1 \pm j)/\sqrt{2}$. Second, the real and imaginary parts in the argument of sine are of the form $m\pi/(4K)$, where the integer m satisfies $-2K \leq m \leq 2K$. This property suggests that we can use either table lookup or multiple-angle formulae of sine to accelerate the computation. Finally, the operation $\mathbf{J}^{\dagger}\text{vec}(\cdot)$ can be implemented readily, since $\mathbf{J}^{\dagger} = \text{diag}(1/w(m))\mathbf{J}^H$ has many zero entries and the weight functions $w(m)$ are integers [16].

Let the finite-snapshot Hermitian Toeplitz matrix based on $\widetilde{\overline{\mathbf{x}}}_{\mathbb{D}}$ be $\widetilde{\overline{\mathbf{R}}}$, as described in the second step in the proposed estimator. It is known that $\widetilde{\overline{\mathbf{R}}}$ is indefinite. Namely, the eigenvalues of $\widetilde{\overline{\mathbf{R}}}$ could

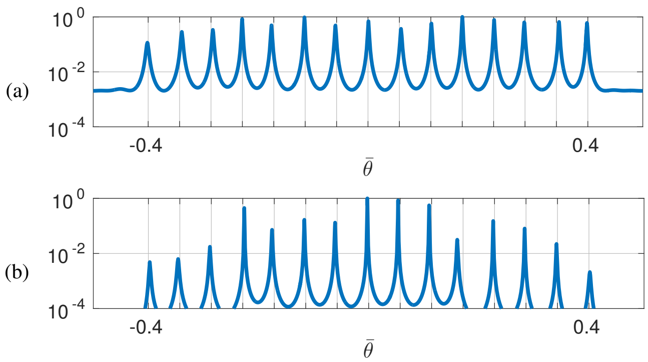

Fig. 2. The SS MUSIC spectra based on the one-bit data $\mathbf{y}_{\mathbb{S}}$ for (a) the nested array ($\text{MSE} = 6.2203 \times 10^{-6}$) and (b) the coprime array ($\text{MSE} = 1.5816 \times 10^{-5}$). There are $D = 15$ uncorrelated and equal-power sources located at $\bar{\theta}_i = -0.4 + 0.8(i-1)/14$ for $i = 1, \ldots, 15$, as indicated by the vertical lines in the plots. Note that this is the case of more sources than sensors ($D = 15 > |\mathbb{S}| = 10$). Number of snapshots $K = 200$ and SNR = 0dB.

be negative. This property does not affect the proposed DOA estimators, since it was shown in [24] that the signal and noise subspace can still be split correctly with indefinite matrices.

5. NUMERICAL RESULTS

In this section, we will study the DOA estimation performance with one-bit quantization using sparse arrays. Let us consider the ULA with 10 sensors, the nested array with $N_1 = N_2 = 5$, and the coprime array with $M = 3$ and $N = 5$. These arrays are also illustrated in Fig. 1. All these arrays have 10 sensors, but the number of identifiable uncorrelated sources is 9 for ULA, 29 for the nested array, and 17 for the coprime array, respectively. The sources are assumed to have equal power. The number of sources D is known to the estimators.

Fig. 2 plots the SS MUSIC spectra *using sparse arrays and one-bit quantized measurements*, as described in Section 4. The parameters are $D = 15$ sources, 0dB SNR and 200 snapshots. The sources are located at $\bar{\theta}_i = -0.4 + 0.8(i-1)/14$ for $i = 1, \ldots, 15$. Here the number of sources is greater than the number of sensors ($D = 15 > |\mathbb{S}| = 10$), so the ULA will not be able to identify the DOAs. It can be observed that in both plots, the peak locations match with the true normalized DOAs, as indicated in vertical lines. Hence, both arrays can resolve all these sources and the nested array have slightly better performance ($\text{MSE} = 6.2203 \times 10^{-6}$) than the coprime array ($\text{MSE} = 1.5816 \times 10^{-5}$). The mean-squared error is defined as $\text{MSE} = \sum_{i=1}^{D} (\widehat{\bar{\theta}}_i - \bar{\theta}_i)^2 / D$ and the estimated normalized DOAs $\widehat{\bar{\theta}}_i$ are based on the root MUSIC algorithm. This example shows that, it is possible to resolve more sources than sensors using either nested arrays or coprime arrays, even from one-bit data.

Fig. 3(a) investigates the dependence of the MSE on SNR, for fewer sources than sensors ($D = 5 < |\mathbb{S}| = 10$). The curves without quantization (solid lines) are obtained using SS MUSIC [14, 15] while the dashed lines show the proposed DOA estimator with one-bit data. First, if the measurements are not quantized, the least MSE is exhibited by the nested array, followed by the coprime array, and finally the ULA. This phenomenon is because the estimation error is more likely to decrease with the size of the difference coarray [14, 15]. Second, for the same array configuration, the one-bit quantized measurements own larger MSE than the unquantized ones. The

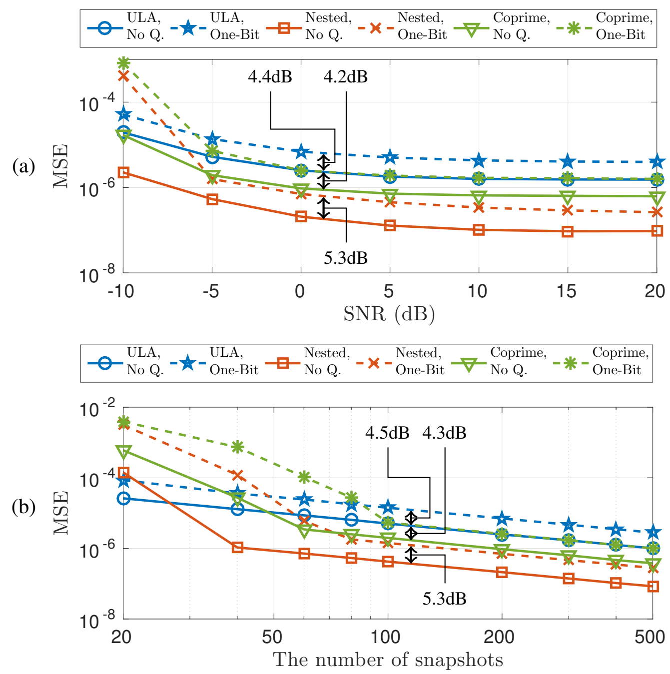

Fig. 3. The dependence of MSE for SS MUSIC and the proposed DOA estimator on (a) SNR with 200 snapshots and (b) snapshots with 0dB SNR for ULA, the nested array, and the coprime array. There are 10 sensors and 5 sources. Solid curves denote the performance without quantization (No Q.) while dashed curves represent the MSEs with one-bit quantization (One-Bit). The source normalized DOAs are $\bar{\theta}_i = -0.4 + 0.2(i-1)$ for $i = 1, \ldots, 5$. Each data point is averaged from 5000 Monte-Carlo runs.

quantization loss, defined as $10 \log_{10}(\text{MSE}_{\text{quantized}} / \text{MSE}_{\text{unquantized}})$, is approximately 4 to 6 dB, which is in accordance with the previous work [13]. Finally, the array configuration plays a very crucial role in MSE. For instance, the coprime array with one-bit quantization (the green dashed line) has similar performance to the ULA *without quantization* (the blue solid curve). Furthermore, for sufficiently large SNR, the nested array with one-bit quantization (the red dashed curve) outperforms the ULA and the coprime array with no quantization. The same phenomenon can also be observed in Fig. 3(b), for sufficient snapshots. Hence, in the context of DOA estimation, the performance improvement due to sparse arrays can compensate the degradation caused by one-bit quantization.

6. CONCLUDING REMARKS

This paper proposed a DOA estimator using one-bit data on sparse arrays. The proposed one-bit quantized sparse arrays typically outperform unquantized ULAs for DOA estimation. Thus, the sparsity of the arrays compensates for the one-bit quantizer, to some extent. Furthermore, as in the case of unquantized sparse arrays, one-bit sparse arrays can continue to identify more sources than sensors.

In the future, it is of considerable interest to analyze the performance of the proposed DOA estimator and the exact condition under which more sources than sensors can be identified, with one-bit measurements.

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