Maximally Economic Sparse Arrays and Cantor Arrays

Chun-Lin Liu and P. P. Vaidyanathan
Dept. of Electrical Engineering, MC 136-93
California Institute of Technology, Pasadena, CA 91125, USA
cl.liu@caltech.edu, ppvnath@systems.caltech.edu

***Abstract*—Sparse arrays, where the sensors are properly placed with nonuniform spacing, are able to resolve more uncorrelated sources than sensors. This ability arises from the property that the difference coarray, defined as the differences between sensor locations, has many more consecutive integers (hole-free) than the number of sensors. In some implementations, it might be preferable that a) the arrays be symmetric, b) that the arrays be *maximally economic*, that is, each sensor be *essential*, and c) that the coarray be hole-free. The essentialness property of a sensor means that if it is deleted, then the difference coarray changes. Existing sparse arrays, such as minimum redundancy arrays (MRA), nested arrays, and coprime arrays do not satisfy these three criteria simultaneously. It will be shown in this paper that Cantor arrays meet all the desired properties mentioned above, based on a comprehensive study on the structure of the difference coarray. Even though Cantor arrays were previously proposed in fractal array design, their coarray properties have not been studied earlier. It will also be shown that the Cantor array has a hole-free difference coarray of size $N^{\log_2 3} \approx N^{1.585}$ where N is the number of sensors. This is unlike the sizes of difference coarrays of the MRA, nested array, coprime array (all $\mathcal{O}(N^2)$), and uniform linear arrays ($\mathcal{O}(N)$)[1].**

***Index Terms*—Symmetric arrays, sparse arrays, hole-free difference coarrays, maximally economic arrays, Cantor arrays.**

I. Introduction

Sparse arrays find useful applications in direction-of-arrival (DOA) estimation, which plays a central role in communication, radio astronomy, and radar [1]–[3]. It was known that for certain sparse arrays, such as minimum redundancy arrays (MRA) [4], nested arrays [5], and coprime arrays [6], it is possible to identify more uncorrelated sources than sensors. This property is due to the fact that the difference coarray, defined as the differences between physical sensor locations, contains a contiguous segment much greater than the number of sensors [4]–[6]. In particular, it is desirable that the difference coarray is *hole-free*, that is, it consists of consecutive integers. Therefore all information on the difference coarray can be exploited [5].

In some implementations, arrays with symmetric geometry are preferred, as they simplify computational complexity [7], facilitate array calibration in presence of mutual coupling [8], [9], and improve DOA estimation performance [10]–[12]. In this paper, we consider symmetric arrays with the further property that each sensor be *essential* in the sense that if it is deleted, the coarray will lose at least one element (thereby compromising DOA performance and identifiability). The essentialness property ensures economy of sensors. Hence such arrays are called *maximally economic arrays*. It is true that maximally economic arrays are less robust to sensor failures, compared to arrays like ULA which have redundant sensors [13]. But in this paper, our focus is economy which is ensured by the *essentialness* property. It turns out that standard arrays such as the ULA, MRA, and nested and coprime arrays do not simultaneously satisfy symmetry and maximal economy. For example, the ULA has many inessential sensors [1] whereas the MRA, nested, and coprime arrays are nonsymmetric [4], [5], [14].

In this paper, we will study the Cantor arrays [15]–[17], which are sparse arrays with a fractal geometry [18], [19]. They are symmetric arrays, with sensor locations specified in closed form. The main contribution of this paper resides in a comprehensive study of the properties of the difference coarrays for the Cantor arrays, while the previous work mainly focused on the array factors [15]–[17]. It will be shown that Cantor arrays are maximally economic. We also derive explicit expressions for the weight functions and show that the difference coarray is hole-free. Moreover, the size of the difference coarrays for the Cantor array with $N = 2^r$ ($r = 0, 1, 2, \dots$) physical sensors is 3^r, which is $N^{\log_2 3} \approx N^{1.585}$. This result is quite distinct from ULA, MRA, nested arrays, and coprime arrays, where ULA has $\mathcal{O}(N)$ elements in the difference coarray and the remaining ones have $\mathcal{O}(N^2)$ elements.

The outline of this paper is as follows. Section II reviews the data model and the design criteria for sparse arrays. Section III proposes the essentialness property for sensor arrays while Section IV defines the Cantor array and studies its difference coarray in detail. Finally, Section V concludes this paper.

II. Preliminaries

Assume that D monochromatic sources illuminate a sensor array, where the sensors are located at $n\lambda/2$. Here n belongs to an integer set $\mathbb{S}$ and λ is the wavelength of the incoming sources. Suppose that the ith source has complex amplitude $A_i \in \mathbb{C}$ and DOA $\theta_i \in [-\pi/2, \pi/2)$. The array output $\mathbf{x}_{\mathbb{S}}$ is modeled as

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathbb{S}} \quad \in \mathbb{C}^{|\mathbb{S}|}, \tag{1}$$

where $\bar{\theta}_i = (\sin\theta_i)/2 \in [-1/2, 1/2)$ is the normalized DOA of the ith source. The steering vector $\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)$ satisfies $\langle \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\rangle_n = e^{j2\pi\bar{\theta}_i n}$, where the bracket notation $\langle\cdot\rangle_n$ represents the sample value on the support location n [20], [21]. The additive noise vector is $\mathbf{n}_{\mathbb{S}}$. It is assumed that the sources

[1]This work was supported in parts by the ONR grants N00014-15-1-2118 and N00014-17-1-2732, the NSF grant CCF-1712633, and the California Institute of Technology.

and noise are zero-mean and uncorrelated. Namely, the expectations $\mathbb{E}[\mathbf{s}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathrm{diag}(p_1, p_2, \ldots, p_D, p_n\mathbf{I})$, where $\mathbf{s} \triangleq [A_1 \quad A_2 \quad \ldots \quad A_D \quad \mathbf{n}_{\mathbb{S}}^T]^T$. The powers of the ith source and the noise vector are denoted by p_i and p_n, respectively.

The covariance matrix of $\mathbf{x}_{\mathbb{S}}$ can be expressed as

$$\mathbf{R}_{\mathbb{S}} = \mathbb{E}[\mathbf{x}_{\mathbb{S}}\mathbf{x}_{\mathbb{S}}^H] = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + p_n\mathbf{I}. \tag{2}$$

Vectorizing $\mathbf{R}_{\mathbb{S}}$ and removing duplicate elements yield

$$\mathbf{x}_{\mathbb{D}} = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{D}}(\bar{\theta}_i) + p_n\mathbf{e}_0 \quad \in \mathbb{C}^{|\mathbb{D}|}, \tag{3}$$

where the vector $\mathbf{e}_0$ satisfies $\langle \mathbf{e}_0 \rangle_m = \delta_{m,0}$ [20], [21]. Here $\delta_{p,q}$ is the Kronecker delta function. The difference coarray is defined as

Definition 1. *Difference coarray.* The difference coarray $\mathbb{D}$ contains the differences between the elements in $\mathbb{S}$, i.e., $\mathbb{D} = \{n_1 - n_2 : \forall\, n_1, n_2 \in \mathbb{S}\}$.

Here the array output $\mathbf{x}_{\mathbb{S}}$ on the physical array is converted into the autocorrelation vector $\mathbf{x}_{\mathbb{D}}$ on the difference coarray. If some properties of the difference coarray are satisfied, then it is possible to identify *more uncorrelated sources than sensors* by using DOA estimators on the autocorrelation vectors [5], [22], [23]. In the following development, some desired properties of the difference coarray will be elaborated.

To begin with, let us define some related quantities. The reversed version of an array $\mathbb{S}$ is defined as $\widehat{\mathbb{S}} = \{\max(\mathbb{S}) + \min(\mathbb{S}) - n : n \in \mathbb{S}\}$. An array is symmetric if $\mathbb{S} = \widehat{\mathbb{S}}$. The central ULA segment of $\mathbb{D}$ is defined by the set $\mathbb{U} \triangleq \{m : \{-|m|, \ldots, -1, 0, 1, \ldots, |m|\} \subseteq \mathbb{D}\}$. The shortest ULA containing $\mathbb{D}$ is denoted by $\mathbb{V} \triangleq \{m : \min(\mathbb{D}) \leq m \leq \max(\mathbb{D})\}$. h is a hole in the difference coarray if $h \in \mathbb{V}$ and $h \notin \mathbb{D}$. A difference coarray $\mathbb{D}$ is said to be *hole-free* if $\mathbb{D} = \mathbb{U} = \mathbb{V}$. The weight function is defined as follows:

Definition 2. The weight function $w(m)$ is the number of sensor pairs with separation m. Namely, $w(m) \triangleq |\{(n_1, n_2) : n_1 - n_2 = m\}|$.

With these quantities, we now move on to some desired design criteria regarding the physical array and the difference coarray:

Criterion 1. *Hole-free difference coarray.* If $\mathbb{D}$ is hole-free, then all the entries in the autocorrelation vector can be utilized directly by algorithms such as coarray MUSIC [5]. If $\mathbb{D}$ is not hole-free, then coarray interpolation has to be done before applying coarray MUSIC, which could increase the overall complexity significantly [24]–[27].

Criterion 2. *Large difference coarray.* It was shown that large difference coarray not only increases the number of resolvable sources [5], [21], [28] but also leads to higher spatial resolution in estimating the DOAs [4], [5], [28]. It is also desirable that the size of the difference coarray $|\mathbb{D}|$ grows much faster than the number of physical array $|\mathbb{S}|$.

Criterion 3. *Symmetric physical array.* As mentioned in Sec. I, symmetric arrays are sometimes preferred [7]–[12].

However, array configurations that satisfy the above three properties simultaneously, have not yet been fully explored. Consider some existing array configurations like ULA, MRA, nested arrays, and coprime arrays. ULAs have hole-free difference coarrays, whose size is only $\mathcal{O}(N)$ [1]. MRAs enjoy the largest hole-free difference coarray, but they do not have explicit sensor locations [4]. Nested arrays can achieve large hole-free difference coarrays (size $\mathcal{O}(N^2)$) with closed forms [5]. Coprime arrays own large difference coarrays ($\mathcal{O}(N^2)$) and closed-form sensor locations, but there are holes in the difference coarray [14]. Furthermore, these arrays, except for ULAs, are all nonsymmetric.

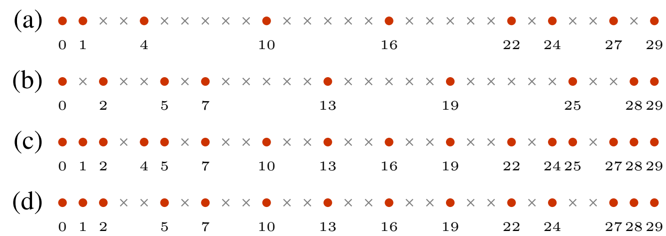

Fig. 1. (a) MRA with 9 sensors; (b) the reversed version of (a); (c) the union of (a) and (b); (d) array configuration after removing 4 and 25 from (c). Here red dots denote sensors while multiplication signs represent empty space.

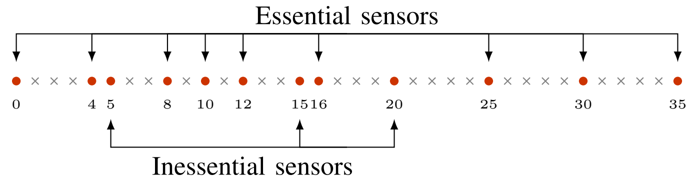

Fig. 2. The coprime array with $M = 4$ and $N = 5$, the essential sensors, and the inessential sensors. The physical sensors are shown in red dots.

In fact, it is quite straightforward to construct arrays satisfying Criteria 1, 2, 3 from existing nonsymmetric arrays with large hole-free difference coarrays. As an example, let us consider the MRA with 9 sensors, as depicted in Fig. 1(a). We first construct its reversed version with respect to the center of the array. The resultant array is shown in Fig. 1(b). The union of Figs. 1(a) and 1(b) results in a new array geometry, as illustrated in Fig. 1(c). It can be shown that 1) Fig. 1(c) is symmetric and 2) it shares the same hole-free difference coarray as Figs. 1(a) and 1(b).

However, some elements in Fig. 1(c) can be removed and the new array configuration still satisfies Criteria 1 to 3. For instance, if the elements 4 and 25 are removed from Fig. 1(c), then the new array, as shown in Fig. 1(d), is symmetric and it can be shown that the new array has the same hole-free difference coarray as Fig. 1(c). In practice, Fig. 1(d) is more cost-effective than Fig. 1(c) since it has fewer number of sensors. This example shows that, apart from Criteria 1, 2, and 3, we need some notion to quantify the importance of each sensor, as we shall propose next.

III. The Essentialness Property

A sensor is said to be essential if the following holds:

Definition 3. Let $\mathbb{S}$ be the physical array and $\mathbb{D}$ be the difference coarray. The sensor located at $n \in \mathbb{S}$ is said to be essential with respect to $\mathbb{S}$ if the difference coarray changes when sensor n is deleted from the array. That is, if $\mathbb{S}' = \mathbb{S} \backslash \{n\}$, then $\mathbb{D}' \neq \mathbb{D}$.

We also say that a sensor is *inessential* if it is not essential. Notice that if sensors n_1 and n_2 are inessential, it does not mean that they can both be deleted without changing

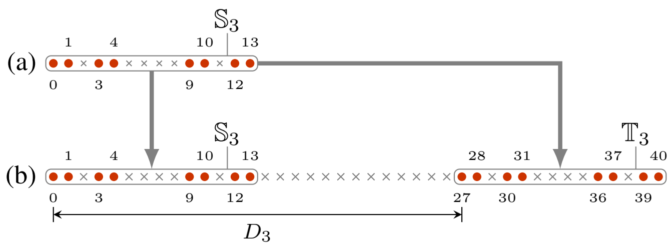

Fig. 3. The relations between the Cantor arrays with (a) $r = 3$ and (b) $r = 4$. Here the dots denote sensors while the multiplication signs represent empty space.

the coarray. But we can remove either one of them without changing the coarray. If each sensor in an array is essential in the above sense, the array is said to be *maximally economic*. It can be shown that MRA and nested arrays are both maximally economic while ULAs and coprime arrays are not.

Fig. 2 demonstrates an example for the essential sensors and the inessential sensors in the coprime array with $M = 4$ and $N = 5$. Here the sensor locations are given by $\{0, M, 2M, \ldots, (N-1)M, N, 2N, \ldots, (2M-1)N\}$ [14]. The essentialness property is examined numerically according to Definition 3. It can be shown that some of the sensors are inessential, such as the ones located at 5, 15, and 20. This means that, for example, removing the element 20 does not influence the difference coarray. This phenomenon is in accordance with what was reported in [29].

Apart from the design criteria in Section II, we will consider another array design criterion regarding the essentialness property:

Criterion 4. *Maximally economic arrays.* This criterion is important if the physical sensors are expensive.

The following lemma shows that the essential sensors are closely related to the weight functions, as follows:

Lemma 1. If $n_1, n_2 \in \mathbb{S}$ and $w(n_1 - n_2) = 1$, then n_1 and n_2 are both essential.

Proof: The statement that $w(n_1 - n_2) = 1$ for $n_1, n_2 \in \mathbb{S}$ implies that (n_1, n_2) is the only sensor pair with separation $m = n_1 - n_2$. If n_1 is removed from $\mathbb{S}$, then the element m is also removed from the difference coarray. Hence n_1 is essential. Similar arguments apply to n_2. ∎

Lemma 1 is useful in identifying the essential sensors. Note that the converse of Lemma 1 is not necessarily true. For instance, consider the array $\mathbb{S} = \{0, 1, 2\}$. It can be shown that the difference coarray is $\mathbb{D} = \{-2, -1, 0, 1, 2\}$, the weight function $w(1) = 2$, but all the sensors are essential.

IV. Symmetric Sparse Arrays

In this section, we will study an array configuration that satisfies Criteria 1, 2, 3, and 4, simultaneously. This array has a very simple and computational tractable recursive definition, which enables us to explicitly write down the expressions for the weight function (Lemma 2). Note that this array is closely related to *Cantor arrays* in fractal array design [15]–[17]. Hence in the following development, the array of interest will be called Cantor arrays, even though our definition is different from those in [15]–[17].

Next, given an array $\mathbb{S}_r$ for a nonnegative integer r, we define the translated array $\mathbb{T}_r \triangleq \{n + D_r : \forall\, n \in \mathbb{S}_r\}$, where $D_r \triangleq 2A_r + 1$, with A_r denoting the aperture of $\mathbb{S}_r$, that is, $A_r \triangleq \max(\mathbb{S}_r) - \min(\mathbb{S}_r)$. With this, we are ready to define a Cantor array:

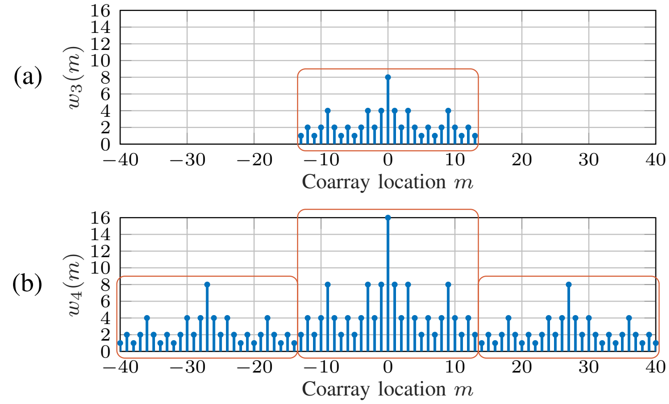

Fig. 4. The weight function for the Cantor arrays with (a) $r = 3$ and (b) $r = 4$.

Definition 4. The Cantor array $\mathbb{S}_r$ is defined recursively as

$$\mathbb{S}_r \triangleq \mathbb{S}_{r-1} \cup \mathbb{T}_{r-1},$$

where $\mathbb{S}_0 \triangleq \{0\}$.

Notice that $\mathbb{S}_r$ has $N = 2^r$ sensors. So, Cantor arrays are defined only for power-of-two N.

The details of Definition 4 are demonstrated in Fig. 3 through a numerical example. Fig. 3(a) depicts the Cantor array with $r = 3$, as denoted by the set $\mathbb{S}_3$. Due to Definition 4, the first half of $\mathbb{S}_4$ is $\mathbb{S}_3$ while the second half is $\mathbb{T}_3$. The amount of translation is given by $D_3 = 2A_3 + 1 = 2 \times 13 + 1 = 27$. It can be seen that $\mathbb{S}_3$ and $\mathbb{S}_4$ are both symmetric arrays.

The arrays in Definition 4 are equivalent to the Cantor array proposed in [15]–[17], with proper amount of translation and scaling. The Cantor arrays in [15]–[17] are built upon the Cantor sets in fractal theory [18], [19]. But here we start with a different definition (Definition 4), which will facilitate the discussion on its coarray properties next.

Compared to the related work [15]–[17], the main contribution of this paper is as follows: The past work on Cantor arrays focused on the array factor and the quantities of interest were the main lobe width and the side lobe levels. In this paper, we focus on the aspect of difference coarrays, with focus on Criteria 1, 2 and 4. To the best of our knowledge, these properties for Cantor arrays have not been investigated in the literature.

Now let us move on to the properties of the Cantor arrays. It can be readily shown that the Cantor arrays are symmetric arrays with $|\mathbb{S}_r| = 2^r$ physical sensors, based on Definition 4. Besides, the weight function of the Cantor array is given by the following lemma, proved in Appendix A:

Lemma 2. For the Cantor array with parameter r in Definition 4, the weight function $w_r(m)$ satisfies

$$w_r(m) = \begin{cases} 2w_{r-1}(m), & \text{if } |m| \leq A_{r-1}, \\ w_{r-1}(m \pm D_{r-1}), & \text{if } |m \pm D_{r-1}| \leq A_{r-1}, \\ 0, & \text{otherwise}, \end{cases} \quad (4)$$

where A_r and D_r are defined as in Definition 4.

Lemma 2 shows that the weight function for the Cantor array $\mathbb{S}_r$ can be recursively constructed from the weight function for $\mathbb{S}_{r-1}$. To give some feelings for Lemma 2, we first utilize Definition 2 to evaluate the weight functions for the Cantor arrays with $r = 3$ and $r = 4$, as depicted in Fig. 4(a) and 4(b), respectively. It can be deduced that the support is from -13 to 13 for $w_3(m)$ and from -40 to 40 for $w_4(m)$. The weight function $w_4(m)$ can be divided into three parts, as marked by rectangles. Then (4) can be verified through the rectangles in Fig. 4. For instance, the weight functions $w_3(10) = 2$ and $w_4(10) = 4$ satisfy the first equation of (4).

Furthermore, Lemma 2 makes it possible to prove the hole-free property (Corollary 1), and the size of the difference coarray (Corollary 2), as we shall show next:

Corollary 1. The difference coarrays of Cantor arrays are hole-free.

Proof: This corollary can be proved by mathematical induction. First it can be readily shown that $\mathbb{S}_0$ has a hole-free difference coarray. Next, assume that $\mathbb{S}_{r-1}$ has a hole-free difference coarray. Then, due to Lemma 2, the difference coarray for $\mathbb{S}_r$ becomes $\mathbb{D}_r = \{m, m \pm D_{r-1} : \forall\, m \in \mathbb{D}_{r-1}\}$. Since $\mathbb{D}_{r-1}$ is hole-free, the terms $m - D_{r-1}$, m, and $m + D_{r-1}$ have consecutive integers from $-(3A_{r-1}+1)$ to $-(A_{r-1}+1)$, from $-A_{r-1}$ to A_{r-1}, and from $A_{r-1}+1$ to $3A_{r-1}+1$, respectively. This means $\mathbb{D}_r$ is also hole-free. ∎

Note that the continuous analogy to Corollary 1 can be found in [30] and the references therein.

Corollary 2. The size of the difference coarrays for the Cantor array with parameter r is 3^r.

Proof: This can also be proved using mathematical induction. It can be readily shown that $|\mathbb{D}_0| = 1$, based on Definition 4. Next, assume that $|\mathbb{D}_{r-1}| = 3^{r-1}$. Due to the proof of Corollary 1, we have $|\mathbb{D}_r| = 2(3A_{r-1}+1)+1 = 3(2A_{r-1}+1) = 3D_{r-1} = 3|\mathbb{D}_{r-1}| = 3^r$. ∎

As a remark, since the Cantor array with parameter r has $|\mathbb{S}_r| = 2^r$ sensors, the size of difference coarray becomes

$$|\mathbb{D}_r| = |\mathbb{S}_r|^{\log_2 3} \approx |\mathbb{S}_r|^{1.585}. \tag{5}$$

Note that the exponent $\log_2 3 \approx 1.585$ in (5) is in fact *the reciprocal of the fractal dimension of the Cantor set* [18], [19]. The relation (5) is quite different from ULA ($|\mathbb{D}| = \mathcal{O}(|\mathbb{S}|)$) and MRA ($|\mathbb{D}| = \mathcal{O}(|\mathbb{S}|^2)$). This means that, for the same large number of elements, the MRA has the largest difference coarray, followed by the Cantor array, and finally the ULA. However, unlike the MRA, the Cantor array has closed-form and symmetric sensor locations.

The last result is the essentialness property of the Cantor array and the proof can be found in Appendix B:

Lemma 3. Cantor arrays are maximally economic.

Finally, let us consider Fig. 4(a) as an example to verify Corollary 1, Corollary 2, and Lemma 3. Since the support of the weight function is the difference coarray, it can be seen that $\mathbb{D}_3$ ranges from -13 to 13. Hence $\mathbb{D}_3$ is hole-free and $|\mathbb{D}_3| = 27 = 3^3$. Furthermore, Fig. 4(a) shows that

$$w_3(13-0) = w_3(12-1) = w_3(10-3) = w_3(9-4) = 1.$$

Hence the all the elements $0, 1, 3, 4, 9, 10, 12, 13$ are essential, due to Lemma 1. This result verifies Lemma 3.

V. Concluding Remarks

This paper considered symmetric and maximally economic sparse arrays with large hole-free difference coarrays. For most of the known sparse arrays, at least one of these three properties is not true. However, Cantor arrays satisfy all these properties. Furthermore, the sensor location in a Cantor array can be recursively specified. We proved these, and also provided closed form expressions for the weight function of the Cantor array. One limitation in Cantor arrays is that the number of sensors N is required to be a power of two. Their difference coarray has size $N^{\log_2 3} \approx N^{1.585}$.

Future research will be directed toward other array geometries that satisfy Criteria 1 to 4 simultaneously, with more general array sizes than powers of two. Another future direction is to study the essentialness property for arbitrary array configurations.

Appendix A
Proof of Lemma 2

The weight function $w_r(m)$ can be expressed as

$$\begin{aligned} w_r(m) &= \left|\left\{(n_1,n_2) \in \mathbb{S}_r^2 : n_1 - n_2 = m\right\}\right| \\ &= \left|\left\{(n_1,n_2) \in \mathbb{S}_{r-1}^2 : n_1 - n_2 = m\right\}\right| \\ &\quad + \left|\left\{(n_1,n_2) \in \mathbb{T}_{r-1}^2 : n_1 - n_2 = m\right\}\right| \\ &\quad + \left|\{(n_1,n_2) \in \mathbb{S}_{r-1} \times \mathbb{T}_{r-1} : n_1 - n_2 = m\}\right| \\ &\quad + \left|\{(n_1,n_2) \in \mathbb{T}_{r-1} \times \mathbb{S}_{r-1} : n_1 - n_2 = m\}\right|, \end{aligned} \tag{6}$$

which is due to $\mathbb{S}_r = \mathbb{S}_{r-1} \cup \mathbb{T}_{r-1}$ in Definition 4. Since every element in $\mathbb{T}_{r-1}$ can be expressed as $n' + D_{r-1}$, where $n' \in \mathbb{S}_{r-1}$, (6) can be written as

$$\begin{aligned} w_r(m) &= \left|\left\{(n_1,n_2) \in \mathbb{S}_{r-1}^2 : n_1 - n_2 = m\right\}\right| \\ &\quad + \left|\left\{(n_1',n_2') \in \mathbb{S}_{r-1}^2 : n_1' - n_2' = m\right\}\right| \\ &\quad + \left|\left\{(n_1,n_2') \in \mathbb{S}_{r-1}^2 : n_1 - n_2' = m + D_{r-1}\right\}\right| \\ &\quad + \left|\left\{(n_1',n_2) \in \mathbb{S}_{r-1}^2 : n_1' - n_2 = m - D_{r-1}\right\}\right| \\ &= 2w_{r-1}(m) + w_{r-1}(m + D_{r-1}) \\ &\qquad + w_{r-1}(m - D_{r-1}). \end{aligned} \tag{7}$$

Since the aperture of the Cantor array with parameter $r-1$ is A_{r-1}, we have, by definition, $w_{r-1}(m) = 0$ if $|m| > A_{r-1}$. Hence, (7) can be simplified as (4).

Appendix B
Proof of Lemma 3

Let the Cantor array with parameter r be denoted by $\mathbb{S}_r = \{s_1, s_2, \ldots, s_N\}$, where $0 = s_1 < s_2 < \cdots < s_N$ and $N = 2^r$. We will first show that the weight function satisfies $w_r(s_{N+1-k} - s_k) = 1$ for $k = 1, 2, \ldots, N$.

First, if $r = 0$, then $\mathbb{S}_0 = \{0\}$ and $w_0(0) = 1$, which holds trivially. Assume $w_r(s_{N+1-k} - s_k) = 1$ holds true for $\mathbb{S}_r$. Then the sensor locations for $\mathbb{S}_{r+1}$ are given by

$$\mathbb{S}_{r+1} = \{s_1, s_2, \ldots, s_N, s_1 + D_r, s_2 + D_r, \ldots, s_N + D_r\}.$$

It can be shown that $s_N < s_1 + D_r < s_2 + D_r < \cdots < s_N + D_r$. Due to Lemma 2, the weight functions for $\mathbb{S}_{r+1}$ satisfy $w_{r+1}((s_N + D_r) - s_1) = w_r(s_N - s_1) = 1$. Similarly, we can show that $w_{r+1}((s_{N+1-k} + D_r) - s_k) = 1$ for $k = 2, 3, \ldots, N$. This means the same result holds true for $\mathbb{S}_{r+1}$.

Next, based on Lemma 1 and the first part of the proof, we have s_k and s_{N+1-k} for $k = 1, 2, \ldots, N$ are both essential, which proves this lemma.

REFERENCES

[1] H. L. Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*. Hoboken, NJ, USA: Wiley, 2002.

[2] R. N. Bracewell, *Radio Astronomy Techniques*. Berlin: Springer, 1962, vol. 54.

[3] M. I. Skolnik, *Introduction to Radar Systems*, 3rd ed. McGraw Hill, 2001.

[4] A. T. Moffet, "Minimum-redundancy linear arrays," *IEEE Trans. Antennas Propag.*, vol. AP-16, no. 2, pp. 172–175, March 1968.

[5] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[6] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.

[7] R. L. Haupt, "Thinned arrays using genetic algorithms," *IEEE Trans. Antennas Propag.*, vol. 42, no. 7, pp. 993–999, July 1994.

[8] B. Friedlander and A. Weiss, "Direction finding in the presence of mutual coupling," *IEEE Trans. Antennas Propag.*, vol. 39, no. 3, pp. 273–284, March 1991.

[9] M. Lin and L. Yang, "Blind calibration and DOA estimation with uniform circular arrays in the presence of mutual coupling," *IEEE Antennas Wireless Propag. Lett.*, vol. 5, no. 1, pp. 315–318, Dec. 2006.

[10] G. Xu, R. H. Roy, and T. Kailath, "Detection of number of sources via exploitation of centro-symmetry property," *IEEE Trans. Signal Process.*, vol. 42, no. 1, pp. 102–112, Jan. 1994.

[11] X. Xu, Z. Ye, Y. Zhang, and C. Chang, "A deflation approach to direction of arrival estimation for symmetric uniform linear array," *IEEE Antennas Wireless Propag. Lett.*, vol. 5, no. 1, pp. 486–489, Dec. 2006.

[12] Z. Ye and X. Xu, "DOA estimation by exploiting the symmetric configuration of uniform linear array," *IEEE Trans. Antennas Propag.*, vol. 55, no. 12, pp. 3716–3720, Dec. 2007.

[13] C. Zhu, W. Q. Wang, H. Chen, and H. C. So, "Impaired sensor diagnosis, beamforming, and DOA estimation with difference co-array processing," *IEEE Sensors J.*, vol. 15, no. 7, pp. 3773–3780, July 2015.

[14] P. P. Vaidyanathan and P. Pal, "Theory of sparse coprime sensing in multiple dimensions," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3592–3608, Aug. 2011.

[15] C. Puente-Baliarda and R. Pous, "Fractal design of multiband and low side-lobe arrays," *IEEE Trans. Antennas Propag.*, vol. 44, no. 5, pp. 730–739, May 1996.

[16] D. H. Werner, R. L. Haupt, and P. L. Werner, "Fractal antenna engineering: the theory and design of fractal antenna arrays," *IEEE Antennas Propag. Mag.*, vol. 41, no. 5, pp. 37–58, Oct. 1999.

[17] D. H. Werner and S. Ganguly, "An overview of fractal antenna engineering research," *IEEE Antennas Propag. Mag.*, vol. 45, no. 1, pp. 38–57, Feb. 2003.

[18] J. Feder, *Fractals*. Springer, 1988.

[19] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed. Wiley, 2005.

[20] C.-L. Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," *IEEE Signal Process. Lett.*, vol. 22, no. 9, pp. 1438–1442, Sept. 2015.

[21] C.-L. Liu and P. P. Vaidyanathan, "Cramér-Rao bounds for coprime and other sparse arrays, which find more sources than sensors," *Digit. Signal Process.*, vol. 61, pp. 43–61, Feb. 2017, special Issue on Coprime Sampling and Arrays. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1051200416300264

[22] S. Pillai, Y. Bar-Ness, and F. Haber, "A new approach to array geometry for improved spatial spectrum estimation," *Proc. IEEE*, vol. 73, no. 10, pp. 1522–1524, Oct. 1985.

[23] Y. Abramovich, D. Gray, A. Gorokhov, and N. Spencer, "Positive-definite Toeplitz completion in DOA estimation for nonuniform linear antenna arrays–Part I: Fully augmentable arrays," *IEEE Trans. Signal Process.*, vol. 46, no. 9, pp. 2458–2471, Sept. 1998.

[24] Y. Abramovich, N. Spencer, and A. Gorokhov, "Positive-definite Toeplitz completion in DOA estimation for nonuniform linear antenna arrays–Part II: Partially augmentable arrays," *IEEE Trans. Signal Process.*, vol. 47, no. 6, pp. 1502–1521, June 1999.

[25] T. E. Tuncer, T. K. Yasar, and B. Friedlander, "Direction of arrival estimation for nonuniform linear arrays by using array interpolation," *Radio Science*, vol. 42, no. 4, Aug. 2007.

[26] C.-L. Liu, P. P. Vaidyanathan, and P. Pal, "Coprime coarray interpolation for DOA estimation via nuclear norm minimization," in *Proc. IEEE Int. Symp. Circuits and Syst.*, May 2016, pp. 2639–2642.

[27] H. Qiao and P. Pal, "Unified analysis of co-array interpolation for direction-of-arrival estimation," in *Proc. IEEE Int. Conf. Acoust., Speech, and Sig. Proc.*, March 2017, pp. 3056–3060.

[28] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the MUSIC algorithm," in *Proc. IEEE Dig. Signal Process. Signal Process. Educ. Workshop*, Jan. 2011, pp. 289–294.

[29] A. Raza, W. Liu, and Q. Shen, "Thinned coprime arrays for DOA estimation," in *Proc. European Signal Proc. Conf.*, Aug. 2017, pp. 415–419.

[30] R. L. Kraft, "What's the difference between Cantor sets?" *The American Mathematical Monthly*, vol. 101, no. 7, pp. 640–650, 1994. [Online]. Available: http://www.jstor.org/stable/2974692