Coprime Coarray Interpolation for DOA Estimation via Nuclear Norm Minimization

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***Abstract*—Coprime arrays, consisting of two uniform linear arrays whose inter-element separations are coprime, can resolve $O(MN)$ sources using only $O(M+N)$ sensors. However, holes in the coarray prevent us from using the full coarray in the MUSIC algorithm for DOA estimation. Through interpolation, it may be possible to use the remaining elements of the coarray to increase the degrees of freedom beyond what is captured in the contiguous ULA section in the coarray. Techniques like positive definite Toeplitz completion, array interpolation, and sparse recovery, manage to include all the information in the coarray, but they demand extra fine-tuned parameters and have individual drawbacks. In this paper, a simple and tractable convex framework via nuclear norm minimization is presented. This approach has no extra tuning parameters and overcomes several undesired issues of other techniques. Numerical examples indicate that, in many instances, the proposed method not only increases the estimation accuracy but also distinguishes more sources than other methods.[1]**

***Index Terms*—Coprime sensor arrays, DOA estimation, nuclear norm, interpolation.**

I. Introduction

Coprime arrays have recently received considerable attention in direction-of-arrival (DOA) estimation problems [1]–[5]. Their major advantage is that $O(MN)$ sources can be identified using only $O(M+N)$ physical sensors, where M and N are a coprime pair of positive integers. Moreover, the closed-form sensor locations are determined simply by two uniform linear arrays (ULAs), whose sensor separations are $M\lambda/2$ and $N\lambda/2$, respectively. Here λ is the wavelength of the incoming signal.

For sparse arrays such as coprime and nested arrays, the coarray MUSIC algorithm is used to estimated DOA [1], [5], [6]. When there are holes in the coarray (as in the case of coprime arrays), techniques such as positive definite Toeplitz completion [7], and array interpolation [8], have been proposed in the past. However, they require extra tuning parameters to work properly. A different approach to include full coarray information is to use convex sparse recovery programs like ℓ_1-minimization or LASSO [9]. But this requires discretization of parameter space into a dense grid, and does not work well for off grid targets.

In this paper, the missing samples or holes are interpolated by nuclear norm minimization which is associated with [10], [11]. This method is not only computationally tractable but also free from predefined dense grids, positive definite requirements, and extra tuning parameters, in comparison to [7]–[9]. Note that our work can be regarded as a matrix completion problem, which aims to recover a matrix based on several known entries [10], [12], [13]. The main difference between our work and [10], [12], [13] is as follows: In this paper, the locations of the known entries are deterministic and related to coprime arrays. By contrast, it is assumed in [10], [12], [13] that the known entries are randomly selected.

Section II reviews sparse array processing. In Section III, a novel coarray interpolation technique is proposed via nuclear norm minimization. Section IV differentiates the proposed approach from several related work [5], [7]–[9], [11]. Section V presents some examples, showing that the proposed method can improve the estimation accuracy and resolve more sources than [5], [7]–[9], [14].

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II. Preliminaries

Assume that D monochromatic and uncorrelated sources impinge on the sensors, located at nd, where n belongs to an integer set $\mathbb{S}$, $d=\lambda/2$ is the inter-element spacing, and λ is the wavelength. For coprime arrays, the set $\mathbb{S}$ is given by

$$\{0, M, \ldots, (N-1)M, N, 2N, \ldots, (2M-1)N\}, \quad (1)$$

where M and N are a coprime pair of positive integers. The sensor outputs can be modeled as

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathbb{S}}. \quad (2)$$

Here A_i and $\bar{\theta}_i$ denote the amplitude and the normalized DOA of the ith source. The normalized DOA is defined as $\bar{\theta} = (d/\lambda)\sin\theta$, where $\theta \in [-\pi/2, \pi/2]$ is the DOA. The steering vector $\mathbf{v}_{\mathbb{S}}(\bar{\theta})$ satisfies $\mathbf{v}_{\mathbb{S}}(\bar{\theta}) = [e^{j2\pi\bar{\theta}n}]_{n\in\mathbb{S}}$. $\mathbf{n}_{\mathbb{S}}$ is the noise term. It is assumed that sources and noise are zero-mean and uncorrelated with each other. Namely, $\mathbb{E}[A_i^* A_j] = \sigma_i^2 \delta_{i,j}$, $\mathbb{E}[A_i^* \mathbf{n}_{\mathbb{S}}] = \mathbf{0}$, and $\mathbb{E}[\mathbf{n}_{\mathbb{S}}\mathbf{n}_{\mathbb{S}}^H] = \sigma^2\mathbf{I}$. σ_i^2 is the power of the ith source, σ^2 is the noise power, and $\delta_{p,q}$ is the Kronecker delta.

According to (2), the covariance matrix of $\mathbf{x}_{\mathbb{S}}$ becomes

$$\mathbf{R}_{\mathbb{S}} = \sum_{i=1}^{D} \sigma_i^2 \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + \sigma^2\mathbf{I}. \quad (3)$$

Reshaping (3) yields the autocorrelation vector $\mathbf{x}_{\mathbb{D}}$:

$$\mathbf{x}_{\mathbb{D}} = \sum_{i=1}^{D} \sigma_i^2 \mathbf{v}_{\mathbb{D}}(\bar{\theta}_i) + \sigma^2\mathbf{e}_0, \quad (4)$$

where $\langle \mathbf{e}_0 \rangle_n = \delta_{n,0}$, and $\langle\cdot\rangle$ is the bracket notation, as defined in [15]. $\mathbb{D}$ is the (difference) coarray of $\mathbb{S}$, defined as

Definition 1 ($\mathbb{D}$, difference coarray). *For a sparse array specified by an integer set $\mathbb{S}$, its difference coarray $\mathbb{D}$ is defined as $\mathbb{D} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{S}\}$.*

The autocorrelation vector $\mathbf{x}_{\mathbb{D}}$ (4), sharing similar expressions as (2), can be regarded as sensor outputs on $\mathbb{D}$. If $\mathbb{D}$ contains a long contiguous ULA segment, which will be denoted by the set $\mathbb{U}$ in Definition 2, then DOAs can be estimated via coarray MUSIC on the autocorrelation evaluated at $\mathbb{U}$ [5]. This approach ensures that for coprime arrays, coarray MUSIC can resolve $O(|\mathbb{U}|) = O(|\mathbb{S}|^2)$ uncorrelated sources using $|\mathbb{S}|$ physical sensors [5]. The following definitions will be useful:

Definition 2 ($\mathbb{U}$). *Let $\mathbb{S}$ denote a sparse array and $\mathbb{D}$ be its difference coarray. The maximum central contiguous ULA segment in $\mathbb{D}$ is $\mathbb{U} = \{m \mid \{-|m|, \ldots, -1, 0, 1, \ldots, |m|\} \subseteq \mathbb{D}\}$.*

Definition 3 ($\mathbb{V}$). *Let $\mathbb{S}$ denote a sparse array and $\mathbb{D}$ be its difference coarray. The shortest ULA containing $\mathbb{D}$ is defined as the integer set $\mathbb{V} = \{m \mid \min(\mathbb{D}) \leq m \leq \max(\mathbb{D})\}$.*

The autocorrelations of sensor output signal evaluated at lags defined by $\mathbb{D}$, $\mathbb{U}$, and $\mathbb{V}$ are denoted by $\mathbf{x}_{\mathbb{D}}$, $\mathbf{x}_{\mathbb{U}}$, and $\mathbf{x}_{\mathbb{V}}$, respectively. *The degree of freedom (DOF)* is the cardinality of $\mathbb{D}$ and *the uniform degree of freedom (uniform DOF)* is the cardinality of $\mathbb{U}$. Coarray MUSIC can exploit the uniform DOF but not the full DOF.

Example 1. *As an example, if $\mathbb{S} = \{0, 1, 4\}$, then*

$$\mathbb{D} = \{-4, -3, -1, 0, 1, 3, 4\}, \qquad \mathbb{U} = \{-1, 0, 1\},$$
$$\mathbb{V} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}.$$

It is evident that $\mathbb{U} \subseteq \mathbb{D} \subseteq \mathbb{V}$. The DOF and the uniform DOF are 7 and 3, respectively. Besides, if $\mathbf{x}_{\mathbb{V}} = [1, 2, 3, 4, 5, 4, 3, 2, 1]^T$, then $\mathbf{x}_{\mathbb{D}} = [1, 2, 4, 5, 4, 2, 1]^T$, and $\mathbf{x}_{\mathbb{U}} = [4, 5, 4]^T$.

In the finite snapshot setup, let $\widetilde{\mathbf{x}}_{\mathbb{S}}(k)$ for $k = 1, \ldots, K$ be K realizations of (2). The covariance matrix of $\mathbf{x}_{\mathbb{S}}$ can be estimated by $\widetilde{\mathbf{R}}_{\mathbb{S}} = \sum_{k=1}^{K} \widetilde{\mathbf{x}}_{\mathbb{S}}(k)\widetilde{\mathbf{x}}_{\mathbb{S}}^H(k)/K$, from which the finite snapshot autocorrelation vector on $\mathbb{U}$, $\widetilde{\mathbf{x}}_{\mathbb{U}}$, can be evaluated as in Definition 3 of [15]. We construct a Hermitian Toeplitz matrix $\widetilde{\mathbf{R}}_{\mathbb{U}}$ satisfying

$$\langle \widetilde{\mathbf{R}}_{\mathbb{U}} \rangle_{n_1, n_2} = \langle \widetilde{\mathbf{x}}_{\mathbb{U}} \rangle_{n_1 - n_2}, \quad \widetilde{\mathbf{R}}_{\mathbb{U}} \in \mathbb{C}^{|\mathbb{U}^+| \times |\mathbb{U}^+|}, \tag{5}$$

where $n_1, n_2 \in \mathbb{U}^+ = \{n \mid n \in \mathbb{U}, n \geq 0\}$. It was shown in [15] that, if the eigenspace is partitioned by the absolute values of the eigenvalues, MUSIC on $\widetilde{\mathbf{R}}_{\mathbb{U}}$ is the same as MUSIC on the spatially smoothed matrix as in [5].

III. Coarray Interpolation Using Nuclear Norm Minimization

In traditional coarray MUSIC including [5] and [15], that part of correlation information in $\mathbb{D}$ which is not also in $\mathbb{U}$ is not used because, as seen from (5), only $\mathbb{U}$ is involved. If the remaining correlation lags (which are not a part of $\mathbb{U}$) are also utilized in MUSIC via interpolation, the estimation error can be reduced and more sources could be identified, compared to [5], [15].

First, the DOF and the uniform DOF for coprime arrays are characterized by the following:

Lemma 1. *The cardinalities of $\mathbb{S}$, $\mathbb{D}$, $\mathbb{U}$, and $\mathbb{V}$ for coprime arrays, as defined in* (1)*, are given by*

$$|\mathbb{S}| = N + 2M - 1, \qquad |\mathbb{D}| = 3MN + M - N,$$
$$|\mathbb{U}| = 2MN + 2M - 1, \qquad |\mathbb{V}| = 4MN - 2N + 1.$$

Proof: The cardinality of $\mathbb{S}$ is trivial under the definition of coprime arrays (1). $|\mathbb{D}|$ and $|\mathbb{U}|$ can be obtained from the coprime array with compressed inter-element spacing (CACIS) in Table I of [2], by substituting $M, \breve{M}$ of [2] with $2M, M$ of this paper. The maximum and minimum elements in $\mathbb{V}$ are $(2M-1)N$ and $-(2M-1)N$, respectively. Therefore, $|\mathbb{V}| = 2(2M-1)N + 1 = 4MN - 2N + 1$. ∎

The uniform DOF $\mathcal{F}$, determines the maximum number of detectable sources by using the algorithm in [2], [5], [15]:

$$(\mathcal{F} - 1)/2 = MN + M - 1 = O(MN). \tag{6}$$

Next, considering the cardinality of the set $\mathbb{D}\backslash\mathbb{U}$ (those in $\mathbb{D}$ but not in $\mathbb{U}$), we obtain $|\mathbb{D}| - |\mathbb{U}| = (M-1)(N-1) = O(MN)$. This is the number of lost freedoms in coarray MUSIC, which is huge for large M and N [5], [15].

To apply coarray MUSIC with all the correlation information in $\mathbb{D}$, we will *interpolate $\widetilde{\mathbf{x}}_{\mathbb{D}}$ on the non-uniform grid $\mathbb{D}$, to $\widetilde{\mathbf{x}}_{\mathbb{V}}$ on the uniform grid $\mathbb{V}$*. But, simple interpolations like spline interpolations [14] to construct $\widetilde{\mathbf{x}}_{\mathbb{V}}$ from $\widetilde{\mathbf{x}}_{\mathbb{D}}$ do not work well. It is because $\widetilde{\mathbf{x}}_{\mathbb{D}}$ originates from autocorrelation functions that have certain structures (4). One structure is that both $\widetilde{\mathbf{x}}_{\mathbb{U}}$ and $\widetilde{\mathbf{x}}_{\mathbb{V}}$ are Hermitian symmetric and they can generate Toeplitz matrices $\widetilde{\mathbf{R}}_{\mathbb{U}}$ and $\widetilde{\mathbf{R}}_{\mathbb{V}}$, respectively. Another structure is that $\widetilde{\mathbf{R}}_{\mathbb{U}}$ and $\widetilde{\mathbf{R}}_{\mathbb{V}}$ own low-rank terms, related to the signal components (Item 4 of Section IV in [15]).

The recovery of missing correlation information in $\widetilde{\mathbf{R}}_{\mathbb{V}}$ can be formulated as a nuclear norm minimization problem. This is conceptually equivalent to producing the interpolated correlation $\widetilde{\mathbf{R}}_{\mathbb{V}}$ from the information contained in $\widetilde{\mathbf{R}}_{\mathbb{D}}$. It was shown in [10], [12], [13] that under certain conditions, nuclear norm is a good convex surrogate for matrix rank (which is non-convex). Nuclear norm minimization can be solved efficiently by semidefinite programming. The interpolation problem in our application is as follows:

$$(\text{P1}): \quad \widetilde{\mathbf{R}}_{\mathbb{V}}^{\star} = \underset{\widetilde{\mathbf{R}}_{\mathbb{V}} \in \mathbb{C}^{|\mathbb{V}^+| \times |\mathbb{V}^+|}}{\arg\min} \|\widetilde{\mathbf{R}}_{\mathbb{V}}\|_* \quad \text{s.t.} \tag{7}$$

$$\widetilde{\mathbf{R}}_{\mathbb{V}} = \widetilde{\mathbf{R}}_{\mathbb{V}}^H, \tag{8}$$

$$\langle \widetilde{\mathbf{R}}_{\mathbb{V}} \rangle_{n_1, n_2} = \langle \widetilde{\mathbf{x}}_{\mathbb{D}} \rangle_{n_1 - n_2}, \tag{9}$$

where $\|\cdot\|_*$ denotes the nuclear norm of a matrix and (9) holds true for all $n_1, n_2 \in \mathbb{V}^+ = \{n \mid n \in \mathbb{V}, n \geq 0\}$, and $n_1 - n_2 \in \mathbb{D}$. According to [10], nuclear norm minimization (7) favors low-rank solutions. The known correlation information on $\mathbb{D}$, $\widetilde{\mathbf{x}}_{\mathbb{D}}$, is fully included in a Hermitian Toeplitz matrix $\widetilde{\mathbf{R}}_{\mathbb{V}}$ as in (8) and (9). The optimal solution to (P1), $\widetilde{\mathbf{R}}_{\mathbb{V}}^{\star}$, contains the interpolated autocorrelation, and can be utilized in computing the coarray MUSIC spectrum readily [15].

Even though (P1) deals with the set $\mathbb{V}$, coarray MUSIC on $\widetilde{\mathbf{R}}_{\mathbb{V}}^{\star}$ cannot always achieve the maximum number of identifiable sources of $\mathbb{V}$, which is $(|\mathbb{V}| - 1)/2 = 2MN - N$. It is because the actual freedom is governed by the non-uniform grid $\mathbb{D}$, as specified in (P1). Coarray interpolation uniquely maps $\widetilde{\mathbf{x}}_{\mathbb{D}}$ to $\widetilde{\mathbf{x}}_{\mathbb{V}}$, which is compatible with coarray MUSIC. This step does not increase the degrees of freedom so that $|\mathbb{V}|$ freedoms are not achievable.

IV. Relation to Other Work

The spatial smoothing MUSIC (SS-MUSIC) [5] combines spatial smoothing with the MUSIC algorithm for $\mathbf{x}_{\mathbb{U}}$. (P1) improves SS-MUSIC in two ways. First, the spatial smoothing step is avoided so that the overall complexity decreases [15]. Second, (P1) includes the information outside $\mathbb{U}$ so that it is likely that (P1) resolves more sources than SS-MUSIC.

The positive-definite Toeplitz matrix completion [7] poses a $\log\det$ maximization problem (Q1) on a positive definite Toeplitz matrix $\mathbf{T}$, in order to construct $\widetilde{\mathbf{x}}_{\mathbb{V}}$ from $\widetilde{\mathbf{x}}_{\mathbb{D}}$. In fact, $\mathbf{T}$ shares the same formulation as $\widetilde{\mathbf{R}}_{\mathbb{V}}$ in (9). However, positive definiteness is indispensable to real-valued $\log\det$ functions [7]. Sometimes, for finite snapshots, positive definiteness makes (Q1) infeasible, so extra manipulations, such as finding the minimum deflective point or adding diagonal loading to $\mathbf{T}$, are vital [7]. On the contrary, the proposed method (P1) *works even for indefinite* $\widetilde{\mathbf{R}}_{\mathbb{V}}$ since the nuclear norm is well-defined and real-valued for any matrix. Furthermore, even if $\widetilde{\mathbf{R}}_{\mathbb{V}}$ is indefinite, Theorem 1 and Corollary 1 in [15] guarantee that the MUSIC spectrum can always be defined from $\widetilde{\mathbf{R}}_{\mathbb{V}}$.

Tuncer *et al*. adapted array interpolation in the coarray to improve DOA estimation performance [8]. Their method jointly updates DOAs and the interpolation matrix with a recursive formula. However, this approach is non-convex, and many parameters, such as the signal power, the noise power, interpolation sectors, and the number of iterations, need to be tuned. By contrast, the convex problem (P1) can be simply solved without knowledge of all these system parameters.

Sparse support recovery techniques can also be used in the coarray domain for DOA estimation. Thus the approach in [9] discretizes potential DOAs into a dense grid and emphasizes sparse solutions via ℓ_1-minimization or LASSO. But the performance depends on the dense grid. If targets are off-grid, the solution might not be sparse, yielding inaccurate DOA estimators. In this paper, we simply solve (P1) which yields low-rank solutions and DOAs are estimated through the MUSIC algorithm, which is free from off-grid issues.

A gridless DOA estimator via low-rank recovery is reported in [11], and considers nuclear norm minimization. As a comparison, (M1) in [11] is summarized as follows:

$$(\mathrm{M1}): \quad \min_{\mathbf{R}} \|\mathbf{R}\|_* \text{ s.t. } \|\mathcal{P}_{smooth}(\widehat{\mathbf{R}}_{yy}) - \mathbf{R}\|_F \le \epsilon. \quad (10)$$

However, (P1) is distinct from (M1):

1) (M1) uses nested arrays, which have hole-free coarrays, so array interpolation is not required. Besides, the purpose of (M1) is denoising, as characterized by ϵ. On the other hand, (P1) considers coprime arrays, which have holes in the coarray. The aim of (P1) is array interpolation so that equailties are enforced in (9).

2) In (M1), the spatial smoothing operator $\mathcal{P}_{smooth}$ acts on the covariance matrix $\widehat{\mathbf{R}}_{yy}$. According to [6], the spatially smoothed matrix $\widetilde{\mathbf{R}}_{ss}$ is evaluated first and $\mathcal{P}_{smooth}(\widehat{\mathbf{R}}_{yy})$ is the positive semidefinite squared root of $\widetilde{\mathbf{R}}_{ss}$. In (P1), $\widetilde{\mathbf{R}}_{\mathbb{V}}$ avoids the spatial smoothing step and can be established readily from $\widetilde{\mathbf{x}}_{\mathbb{D}}$ [15].

3) $\widetilde{\mathbf{R}}_{\mathbb{V}}$ admits simple linear constraints (9). If (P1) is formulated in terms of the spatially smoothed matrix $\widetilde{\mathbf{R}}_{ss}$, the linear constraints (9) become nonlinear equality constraints since $\widetilde{\mathbf{R}}_{ss} = \widetilde{\mathbf{R}}_{\mathbb{V}}^2$ [5], [6], [15]. Hence the overall problem is non-convex with respect to $\widetilde{\mathbf{R}}_{ss}$.

4) The matrix $\mathbf{R}$ in (M1) might not be Hermitian so that, later on, a MUSIC-like spectrum is defined by the singular value decomposition of $\mathbf{R}$ [11]. But, $\widetilde{\mathbf{R}}_{\mathbb{V}}$ in (P1) is restricted to Hermitian matrices, which not only accelerates the convex solvers but also guarantees that the eigen-decomposition remains applicable to MUSIC.

V. Numerical Examples

In this section, a coprime array with $M = 3$ and $N = 5$ is considered. 10 sensors are located at 0, 3, 5, 6, 9, 10, 12, 15, 20, and 25. Lemma 1 suggests that $|\mathbb{S}| = 10$, $|\mathbb{D}| = 43$, $|\mathbb{U}| = 35$, and $|\mathbb{V}| = 51$. The maximum number of identifiable sources using coarray MUSIC on $\widetilde{\mathbf{x}}_{\mathbb{U}}$ is 17, as in (6).

The first example in Fig. 1 demonstrates one instance where our approach (P1) exhibits the best performance. Two uncorrelated and equal-power sources are located at $\bar{\theta}_1 = -0.0045$ and $\bar{\theta}_2 = 0.0045$. The SNR is 0 dB and there are 500 snapshots to estimate correlations using sample averages. For the same autocorrelation vector $\widetilde{\mathbf{x}}_{\mathbb{D}}$, six different DOA estimation methods are used: spatial smoothing MUSIC (SS-MUSIC) [5], coarray LASSO (Co-LASSO) (15) in [9], spline interpolation on $\widetilde{\mathbf{x}}_{\mathbb{D}}$ [14], iterative CA-AI (ICA-AI) [8], positive definite Toeplitz completion [7], and the proposed method (i.e. solving (P1)). For Co-LASSO (Eq. (15) in [9]), we choose $h = 0.2$. The MUSIC spectrum is plotted in Fig. 1 for all methods. For Co-LASSO, the solution $\mathbf{x}$ (with grid size 1024) is plotted in place of the MUSIC spectrum. The ICA-AI plot uses 10 iterations. It can be inferred from Fig. 1 that, almost every method, except Co-LASSO, exhibits two peaks around the true DOAs. Co-LASSO has relatively large error, since targets are off-grid. The root-mean-squared error (RMSE) is defined as $E = (\sum_{i=1}^{D}(\hat{\bar{\theta}}_i - \bar{\theta}_i)^2/D)^{1/2}$, where $\hat{\bar{\theta}}_i$ denotes the estimated normalized DOA of the ith source and $\bar{\theta}_i$ is the true normalized DOA. It can be seen that our proposed method indeed decreases DOA estimation error in this case.

Fig. 2 shows another example with $D = 19$ (which exceeds the limit for SS-MUSIC, which is 17 from (6)). The sources are assumed uncorrelated with equal power. These sources are located at $\bar{\theta}_i = -0.4 + 0.8(i-1)/18$ for $1 \le i \le 19$. The remaining parameters are the same as those in Fig. 1. In this setting, positive definite Toeplitz completion [7] is unable to produce a MUSIC spectrum since the associated optimization problem is infeasible. In Co-LASSO and ICA-AI, false peaks are located around $\bar{\theta} = -0.5$ and $\bar{\theta} = 0.45$, respectively. Spline interpolation misses the target around $\bar{\theta} = 0.3$. The proposed approach is able to generate a clean MUSIC spectrum without false peaks or missing targets, for this particular instance. Our proposed method is therefore very promising for the task of identifying more sources than what is done by SS-MUSIC (Eq. (6)).

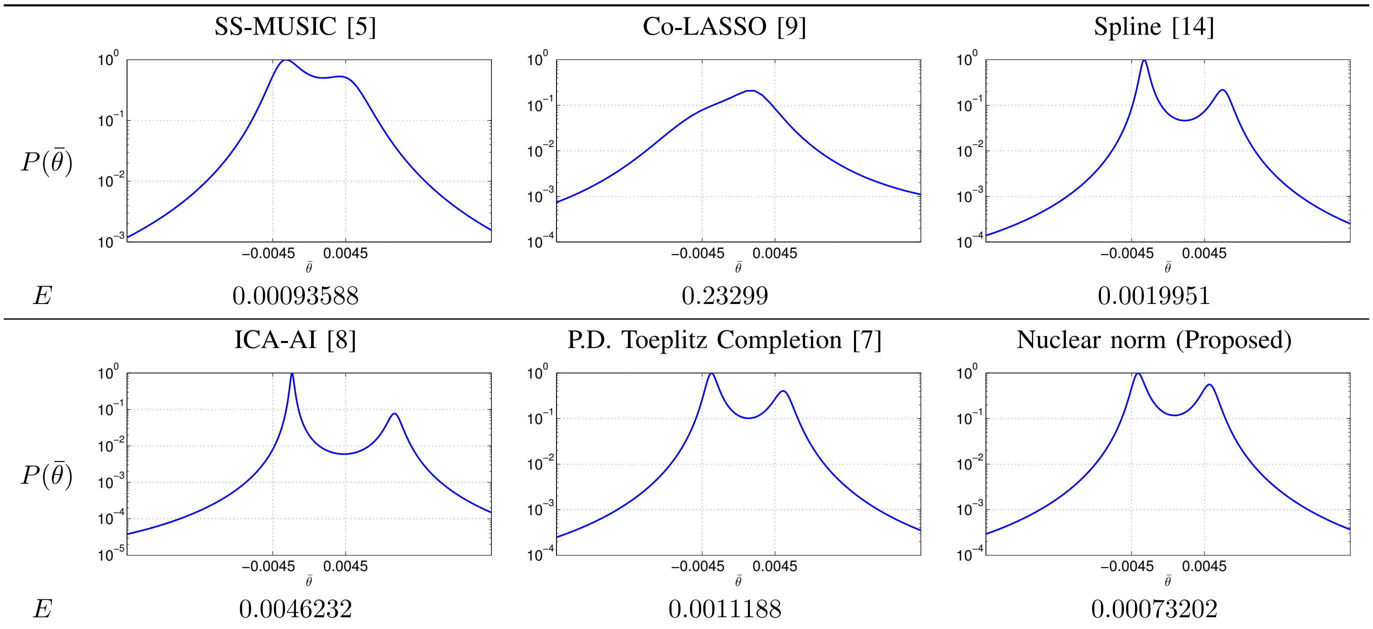

Fig. 1. The spectrum $P(\bar{\theta})$ for various interpolation methods. There are $D = 2$ uncorrelated and equal-power sources with normalized DOAs $\bar{\theta}_1 = -0.0045$, $\bar{\theta}_2 = 0.0045$. The SNR is 0 dB and there are 500 snapshots.

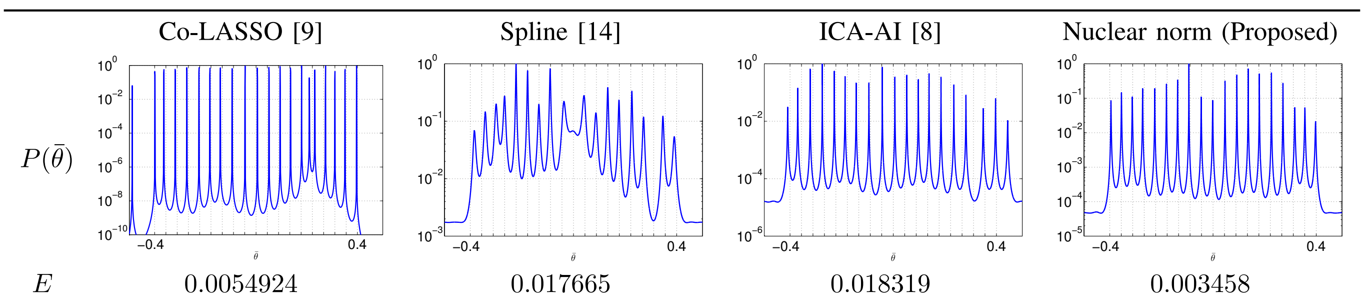

Fig. 2. The spectrum $P(\bar{\theta})$ for various interpolation methods. The number of sources D is 19, exceeding the limit achievable with $\mathbb{U}$ (Eq. (6)) which is 17. These sources are uncorrelated, equal-power, and located at $\bar{\theta}_i = -0.4 + 0.8(i-1)/18$ for $1 \leq i \leq 19$. The SNR is 0 dB and there are 500 snapshots.

VI. Concluding Remarks

A novel coarray interpolation using nuclear norm minimization for coprime arrays was proposed. For many instances, our method is capable of identifying more sources than that in [5]. It also lowers the estimation error in comparison to some related work [5], [7]–[9], [14]. In the future, it will be interesting to incorporate the matrix denoising idea into the interpolation approach.

References

[1] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb 2011.

[2] S. Qin, Y. Zhang, and M. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Process.*, vol. 63, no. 6, pp. 1377–1390, March 2015.

[3] Z. Tan, Y. Eldar, and A. Nehorai, "Direction of arrival estimation using co-prime arrays: A super resolution viewpoint," *IEEE Trans. Signal Process.*, vol. 62, no. 21, pp. 5565–5576, Nov 2014.

[4] K. Adhikari, J. R. Buck, and K. E. Wage, "Extending coprime sensor arrays to achieve the peak side lobe height of a full uniform linear array," *EURASIP J. Adv. Signal Process.*, vol. 2014, no. 1, 2014.

[5] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the MUSIC algorithm," in *Proc. IEEE Digit. Signal Process. Signal Process. Educ. Workshop*, Jan 2011, pp. 289–294.

[6] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug 2010.

[7] Y. Abramovich, N. Spencer, and A. Gorokhov, "Positive-definite Toeplitz completion in DOA estimation for nonuniform linear antenna arrays - Part II: Partially augmentable arrays," *IEEE Trans. Signal Process.*, vol. 47, no. 6, pp. 1502–1521, Jun 1999.

[8] T. E. Tuncer, T. K. Yasar, and B. Friedlander, "Direction of arrival estimation for nonuniform linear arrays by using array interpolation," *Radio Science*, vol. 42, no. 4, 2007.

[9] P. Pal and P. P. Vaidyanathan, "Pushing the limits of sparse support recovery using correlation information," *IEEE Trans. Signal Process.*, vol. 63, no. 3, pp. 711–726, Feb 2015.

[10] B. Recht, M. Fazel, and P. A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM Review*, vol. 52, no. 3, pp. 471–501, 2010.

[11] P. Pal and P. P. Vaidyanathan, "A grid-less approach to underdetermined direction of arrival estimation via low rank matrix denoising," *IEEE Signal Process. Lett.*, vol. 21, no. 6, pp. 737–741, June 2014.

[12] E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Foundations of Computational Mathematics*, vol. 9, no. 6, pp. 717–772, 2009.

[13] D. Donoho and M. Gavish, "Minimax risk of matrix denoising by singular value thresholding," *Ann. Statist.*, vol. 42, no. 6, pp. 2413–2440.

[14] M. Unser, "Splines: a perfect fit for signal and image processing," *IEEE Signal Process. Mag.*, vol. 16, no. 6, pp. 22–38, Nov 1999.

[15] C.-L. Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," *IEEE Signal Process. Lett.*, vol. 22, no. 9, pp. 1438–1442, Sept 2015.