A General Form of 2D Fourier Transform Eigenfunctions

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Abstract

In this paper, the general form of the two-dimensional Fourier transform (2D FT) eigenfunctions is discussed. It is obtained from the linear combination of the 2D separable Hermite Gaussian functions (SHGFs). For example, the rotated Hermite Gaussian functions (RHGFs) for the rotated coordinate and the Laguerre Gaussian functions (LGFs) for the polar coordinate are two special cases of the general form. With the aid of the general form, we can achieve these 2D functions with perfect orthogonality. Finding the combination coefficients is equivalent to the multinomial expansion problem. Therefore, we can apply the fast Fourier transform and some recurrence relations to the coefficients. The computation cost is much less than the close-form coefficients, which is associated with the Jacobi polynomials.

Known Eigenfunctions of the FT

- HGFs: $h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} H_n(x) e^{-x^2/2}$.
- SHGFs: $h_{m,n}(x,y) = h_m(x) h_n(y)$.
- RHGFs: $h_{m,n}(\alpha; x, y) = h_{m,n}(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha)$.
- LGFs: $l_{m,n}(r,\theta) = N_{p,l} r^l L_p^l(r^2) e^{-r^2/2} e^{-jl\theta}$, where $p = \min\{m,n\}$, $l = |m-n|$, $N_{p,l}$ is the normalization factor, and $L_n^m(\cdot)$ is the associated Laguerre polynomial.

The General Form

Assume that the eigenfunction, $\psi^D(x,y)$, is composed of a two-variable polynomial with degree D, and a Gaussian function. We want it to be the eigenfunction and orthonormal.

$$\psi^D(x,y) = \sum_{p=0}^{D} c^D_{p,D-p} h_{p,D-p}(x,y), \tag{1}$$

where $\left[c^D_{0,D}, c^D_{1,D-1}, \ldots, c^D_{D,0}\right]^T$ is a $(D+1)$-dim unitary vector.

- SHGFs $h_{m,n}(x,y)$, $D = m+n$, and $c^{m,n}_{p,q} = \delta[p-m, q-n]$.
- RHGFs $h_{m,n}(\alpha; x, y)$, $D = m+n = p+q$ [1, 4]

$$c^{m,n}_{p,q}(\text{RHGF}) = \sqrt{\frac{p!q!}{m!n!}} (\sin\alpha)^{m-p} (\cos\alpha)^{n-p} P_p^{(m-p,n-p)}(\cos 2\alpha),$$

  where $P_n^{(\alpha,\beta)}(\cdot)$ is the Jacobi polynomial.
- LGFs $l_{m,n}(r,\theta)$, , $c^{m,n}_{p,q}(\text{LGF}) = j^p c^{m,n}_{p,q}(\text{RHGF})\big|_{\alpha=\pi/4}$ [1, 2, 4].

The Fast Coefficient Computation

- In [4], the 2-D separable Hermite polynomial is

$$H_{m,n}(x,y) = \left[2^{m+n} e^{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)}\right](x^n y^m). \tag{2}$$

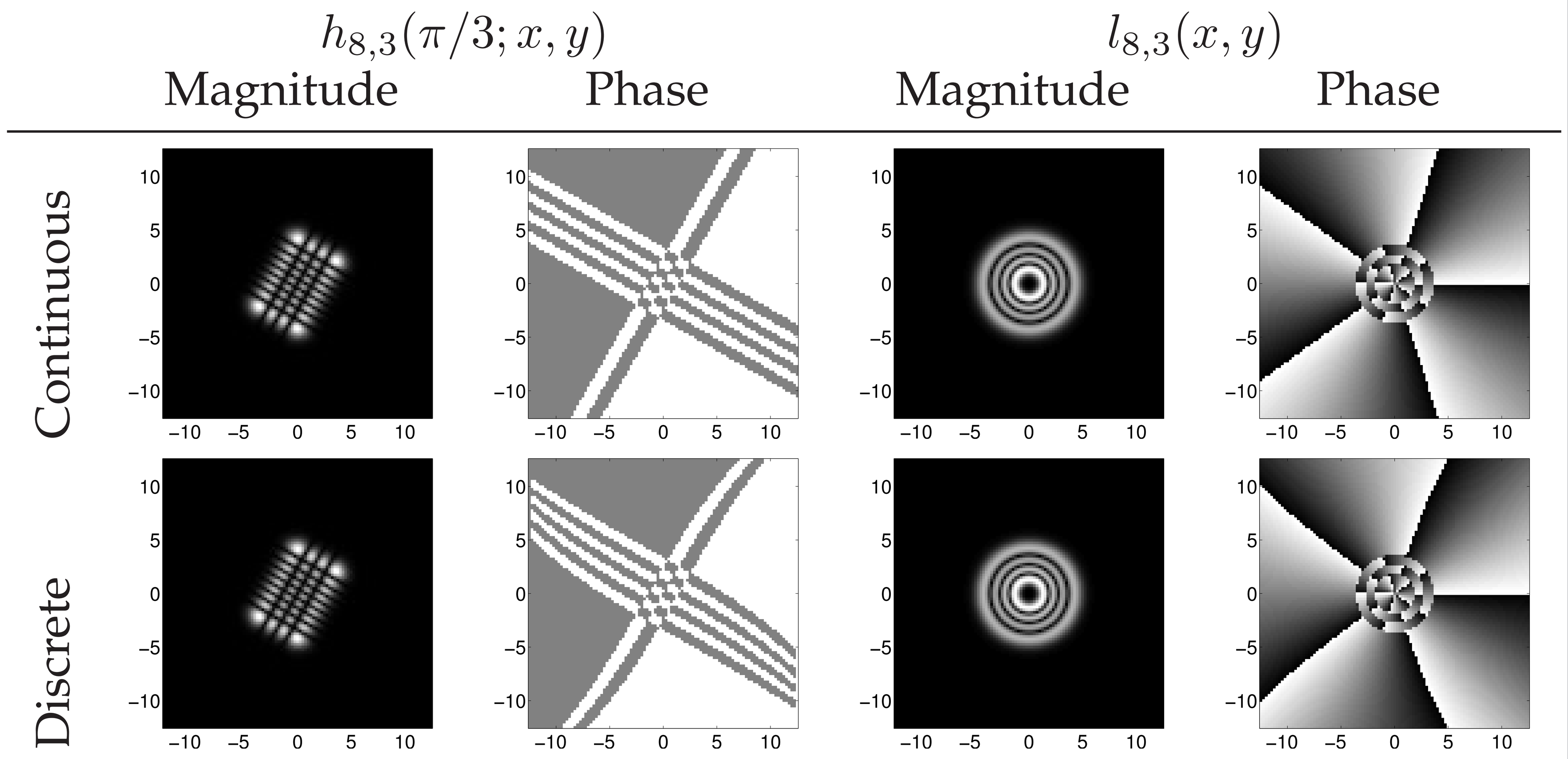
- Apply coordinate transformation, $(x,y) \xrightarrow{U} (x', y')$

$$H_{m,n}(U; x, y) = 2^{m+n} e^{-\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)} (x')^m (y')^n, \tag{3}$$

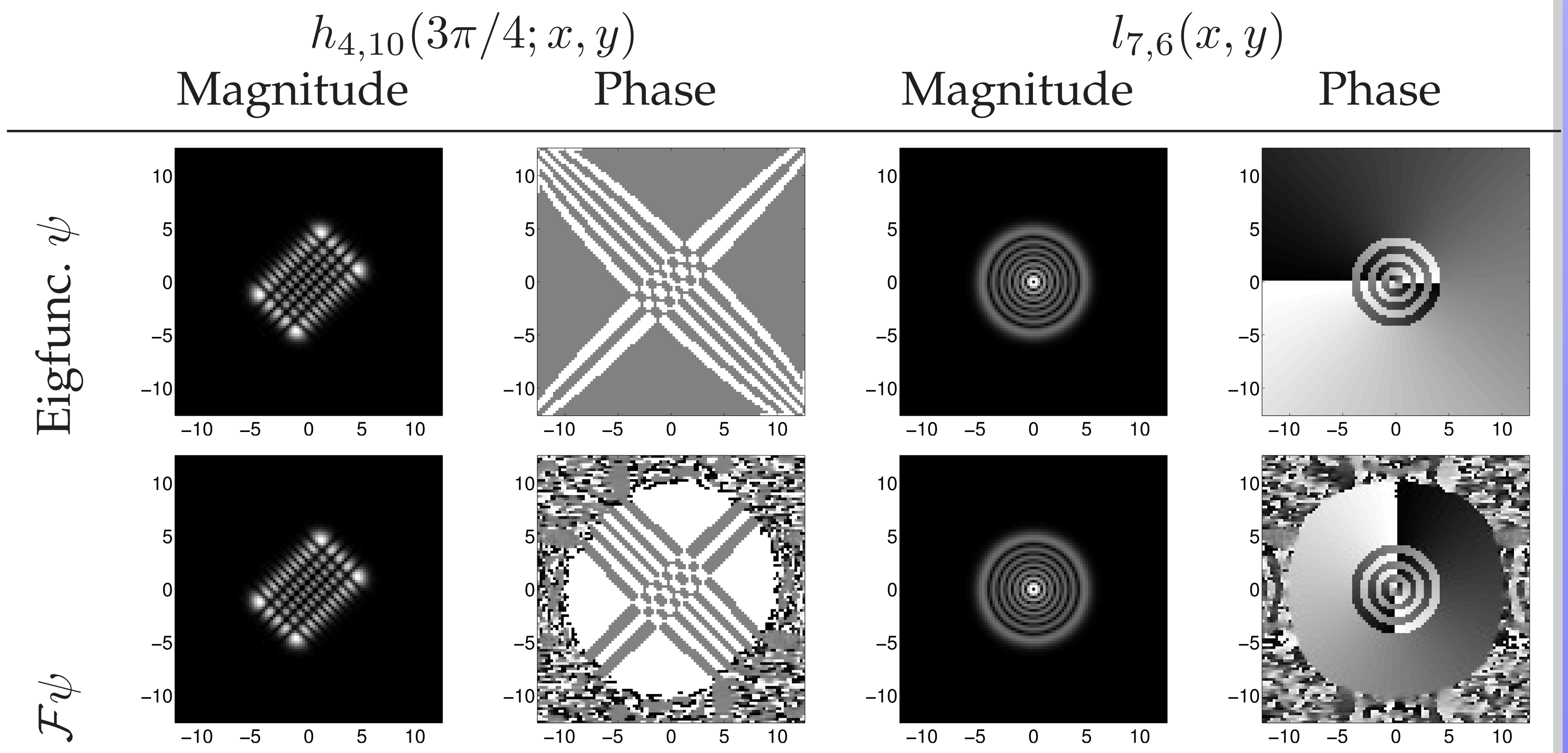
- Expanding $(x')^m (y')^n$ in terms of x, y yields $c^{m,n}_{p,q}(\text{RHGF})$.
- Direct polynomial expansion has complexity $O\left((D+1)^2\right)$ while the convolution using the FFT has complexity $O\left((D+1)\log(D+1)\right)$.

Simulation Results

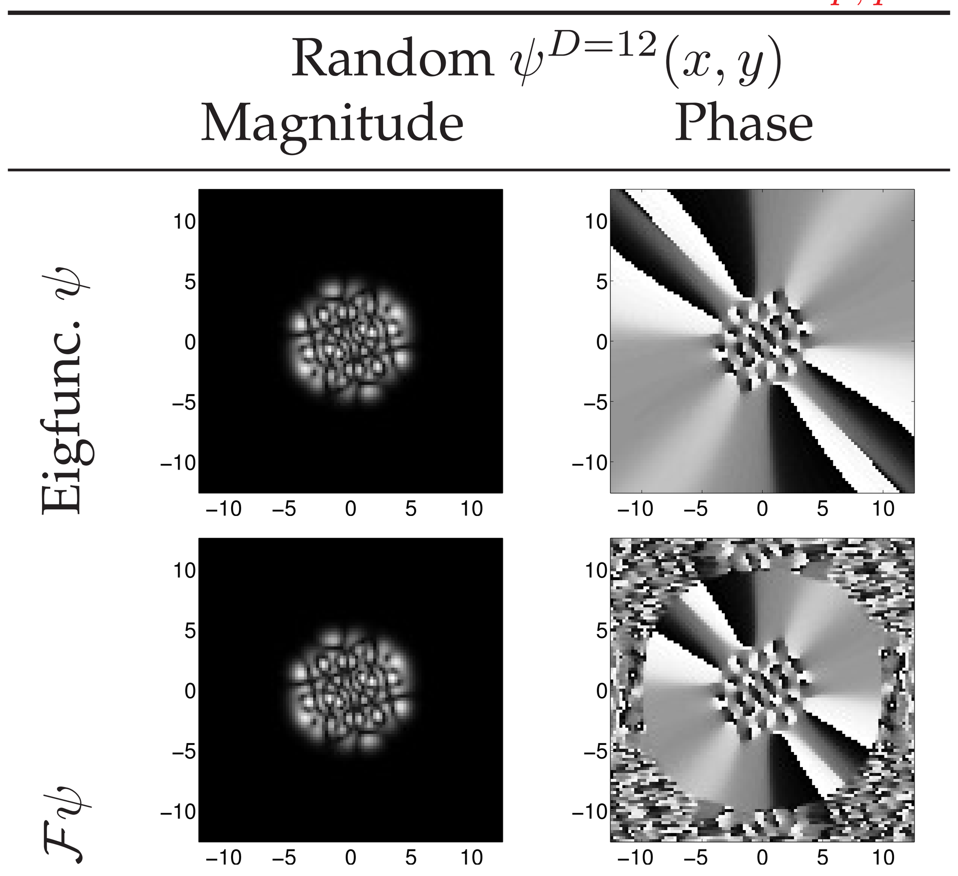
Simulation 1: Accuracy comparison



Simulation 2: RHGFs and LGFs are eigenfunctions of the DFT



Simulation 3: The eigenfunction with random $c^D_{p,q}$



- In [3], we solve 1-D discrete HGFs with number of discrete points to be $N = 101$ and sampling interval $\Delta_x = \Delta_y = \sqrt{2\pi/N}$.
- The discrete functions are orthonormal while the continuous samples are not.
- The phase error on the boundaries is due to small magnitudes below the order of 10^{-16}.

Applications

- Discrete image rotation without interpolation.
- Analysis of rotated or circular symmetric objects on Cartesian grids.
- 2-D discrete fractional Fourier transforms in different coordinate systems.

References

[1] E. G. Abramochkin and V. G. Volostnikov. "Generalized Hermite-Laguerre-Gauss Beams". In: *Physics of Wave Phenomena* 18 (2010), pp. 14–22.

[2] M. W. Beijersbergen et al. "Astigmatic laser mode converters and transform of orbital angular momentum". In: *Opt. Comm.* 96 (1993), pp. 123–132.

[3] F. A. Grünbaum. "The eigenvectors of the discrete Fourier transform: A version of the Hermite functions". In: *J. Math. Anal. Appl.* 88 (1982), pp. 355–363.

[4] A Wünsche. "General Hermite and Laguerre two-dimensional polynomials". In: *J. Phys. A: Math. Gen.* 33 (2000), pp. 1603–1629.