Coprime Arrays and Samplers for Space-Time Adaptive Processing

Chun-Lin Liu
cl.liu@caltech.edu

P. P. Vaidyanathan
ppvnath@systems.caltech.edu

Digital Signal Processing Group
Electrical Engineering
California Institute of Technology

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Outline

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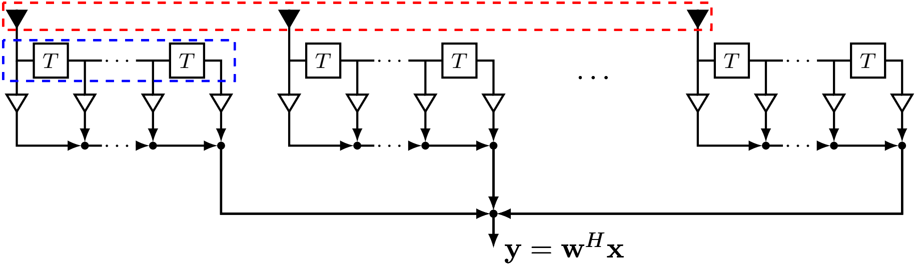
Space-Time Adaptive Processing[1]



- spatial arrays + time sampling.
- STAP resolves the direction-of-arrival (DOA) information and the Doppler frequency jointly.
- Distinguish at most $O(\#$ of sensors $\times$ $\#$ of taps$)$ sources

[1] J. R. Guerci, *Space-Time Adaptive Processing for Radar*. Artech House, 2003.

Different Estimators

- Power spectrum density (PSD)[2],
- Minimum variance distortionless response (MVDR)[3],
- MUSIC[4],
- ESPRIT[5],
- compressive joint angular-frequency power spectrum estimation[6],
- etc...
- Question: Can we estimate more sources than the existing methods?

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[2] R. Klemm, *Space-Time Adaptive Processing: Principles and Applications*. IEEE Press, 1998.

[3] J. Capon, "High-resolution frequency-wavenumber spectrum analysis," *Proc. IEEE*, vol. 57, no. 8, pp. 1408–1418, 1969.

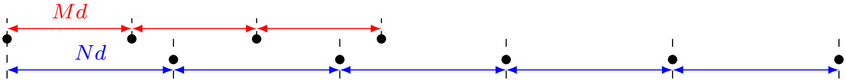
[4] R. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, no. 3, pp. 276–280, 1986.

[5] R. Roy and T. Kailath, "Esprit-estimation of signal parameters via rotational invariance techniques," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 37, no. 7, pp. 984–995, 1989.

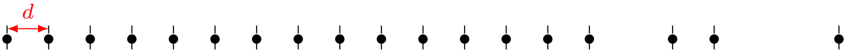
[6] D. D. Ariananda and G. Leus, "Compressive joint angular-frequency power spectrum estimation," in *Proc. the 21st European Signal Processing Conference (EUSIPCO 2013)*, Marrakech, Morocco, 2013.

Coprime Arrays[7]

Physical array:



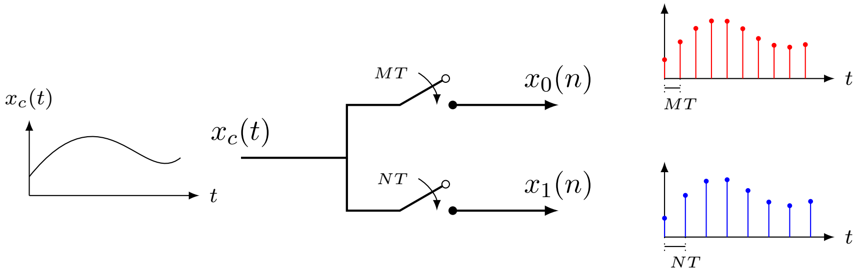
Difference coarray (non-negative part):



- M and N are a coprime pair of integers.
- Union of two uniform linear arrays with interelement spacing Md and Nd, respectively.
- Coprime arrays identify up to $O(MN)$ sources using $O(M+N)$ sensors.

[7] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, 2011.

Coprime Samplers in Time[8]



- Non-uniform samples of $x_c(t)$ ($x_0(n)$ and $x_1(n)$) suffice to compute uniform samples $R_c(nT)$, which is the autocorrelation function of $x_c(t)$.
- Reduced sample rate: from $\frac{1}{T}$ to $\frac{1}{MT} + \frac{1}{NT}$

[8] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, 2011.

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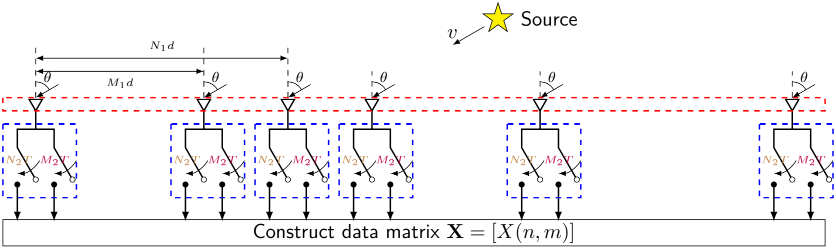
Coprime JADE



- Coprime arrays with parameter M_1, N_1: located at nd, where

$$n \in \mathbb{S}_s = \{0, M_1, \ldots, (N_1 - 1)M_1, N_1, 2N_1, \ldots, (2M_1 - 1)N_1\}.$$

- Coprime samplers with parameter M_2, N_2: samples at $t = mT$, where

$$m \in \mathbb{S}_t = \{0, M_2, \ldots, (N_2 - 1)M_2, N_2, 2N_2, \ldots, (2M_2 - 1)N_2\}.$$

The Data Model (Matrix forms)

The data matrix $\mathbf{X}$ can be modeled as

$$\mathbf{X} = \sum_{i=1}^{D} A_i \mathbf{v}_s\left(\bar{\theta}_i\right) \mathbf{v}_t^T\left(\bar{f}_i\right) + \mathbf{N}, \quad (1)$$

where

- A_i: The complex amplitude of the ith source.
- $\mathbf{v}_s\left(\bar{\theta}_i\right) = \left[e^{j2\pi\bar{\theta}_i n}\right]_{n\in\mathbb{S}_s}$: Spatial steering vectors.
- $\mathbf{v}_t\left(\bar{f}_i\right) = \left[e^{j2\pi\bar{f}_i m}\right]_{m\in\mathbb{S}_t}$: Temporal steering vectors.
- $\bar{\theta}_i = \frac{d}{\lambda}\sin\theta_i$: Normalized DOA. θ_i: DOA.
- $\bar{f}_i = \frac{T}{\lambda}v_i$: Normalized Doppler frequency. v_i: radial velocity.
- $\mathbf{N}$: Additive noise.

The Data Model (Vector forms)

Vectorizing $\mathbf{X}$ gives

$$\mathbf{x} = \mathrm{vec}\left(\mathbf{X}\right) = \sum_{i=1}^{D} A_i \mathbf{v}_{s,t}\left(\bar{\theta}_i, \bar{f}_i\right) + \mathbf{n}. \tag{2}$$

- $\mathbf{v}_{s,t}\left(\bar{\theta}_i, \bar{f}_i\right) = \mathbf{v}_t\left(\bar{f}_i\right) \otimes \mathbf{v}_s\left(\bar{\theta}_i\right)$: Space-time steering vectors.
- $\otimes$: Kronecker products.
- Convert (2) to the coarray-lag domain.

Representation in the Coarray-Lag Domain

- Statistical assumptions:
  - $A_1, \ldots, A_D, \mathbf{n}$ are zero mean, uncorrelated random variables/vectors.
  - For $i = 1, \ldots, D$, $\mathbb{E}\left[|A_i|^2\right] = \sigma_i^2$, $\mathbb{E}\left[\mathbf{n}\mathbf{n}^H\right] = \sigma^2\mathbf{I}$.
- The covariance matrix of $\mathbf{x}$:

$$\mathbf{R}_\mathbf{x} = \mathbb{E}\left[\mathbf{x}\mathbf{x}^H\right] = \sum_{i=1}^{D} \sigma_i^2 \underbrace{\mathbf{v}_{s,t}\left(\bar{\theta}_i, \bar{f}_i\right)\mathbf{v}_{s,t}^H\left(\bar{\theta}_i, \bar{f}_i\right)}_{\left(\mathbf{v}_t(\bar{f}_i)\mathbf{v}_t^H(\bar{f}_i)\right)\otimes\left(\mathbf{v}_s(\bar{\theta}_i)\mathbf{v}_s^H(\bar{\theta}_i)\right)} + \sigma^2\mathbf{I}.$$

- $\mathbf{R}_\mathbf{x}$ can be reshaped into another matrix $\mathbf{Z}$:

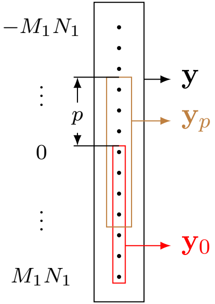
$$\mathbf{Z} = \sum_{i=1}^{D} \sigma_i^2 \mathbf{w}_s\left(\bar{\theta}_i\right)\mathbf{w}_t^T\left(\bar{f}_i\right) + \sigma^2\mathbf{e}_1\mathbf{e}_2^T. \quad (3)$$

  - $\mathbf{w}_s(\bar{\theta}_i)$: Spatial steering vector over the different set of $\mathbb{S}_s$ (Coarrays).
  - $\mathbf{w}_t(\bar{f}_i)$: Temporal steering vector over the difference set of $\mathbb{S}_t$ (Lags).

Spatial Smoothing

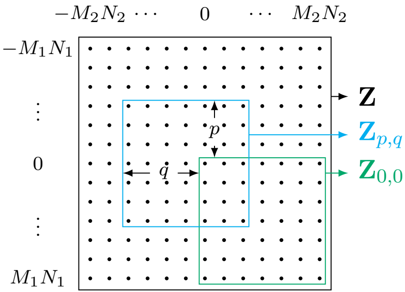
- Establish a full-rank matrix $\mathbf{R}_{ss}$ from $\mathbf{Z}$ and then apply the MUSIC algorithm.
- How to perform spatial smoothing over the matrix $\mathbf{Z}$?

1D Spatial Smoothing



$\mathbf{R}_{ss} \propto \sum_p \mathbf{y}_p\mathbf{y}_p^H$

2D Spatial Smoothing



Remarks

- Coprime JADE is similar to 2D coprime arrays.
  - (DOA, Doppler) $\equiv$ (Azimuth, Elevation).
  - Coprime JADE: Fixed sensor locations and the same sampling pattern is applied to each sensor.
  - 2D coprime arrays: Design over a nonseparable lattice.
- Let $\mathbf{R}_{ss} = \begin{bmatrix}\mathbf{U}_s & \mathbf{U}_n\end{bmatrix} \mathbf{\Lambda} \begin{bmatrix}\mathbf{U}_s & \mathbf{U}_n\end{bmatrix}^H$ as the eigen-decomposition, where $\mathbf{U}_n$ represents the *noise subspace*. The MUSIC spectrum is

$$P_{MUSIC}\left(\bar{\theta}, \bar{f}\right) = \frac{1}{\left\|\mathbf{U}_n^H \tilde{\mathbf{w}}_{s,t}\left(\bar{\theta}, \bar{f}\right)\right\|_2^2},$$

where $\bar{\theta}, \bar{f} \in [-1/2, 1/2)$ and $\|\cdot\|_2$ denotes Euclidean norms of vectors.

The Number of Identifiable Sources[9]

Theorem

Consider distinct sources $S = \left\{\left(\bar{\theta}_i, \bar{f}_i\right)\right\}_{i=1}^{D}$ *where* $D \leq M_1 N_1 M_2 N_2$. *Assume that*

- *the set* $\left\{\bar{\theta}_i\right\}_{i=1}^{D}$ *takes at most* $M_1 N_1$ *distinct values and*
- $\left\{\bar{f}_i\right\}_{i=1}^{D}$ *contains at most* $M_2 N_2$ *distinct values.*

Then, $P_{MUSIC}\left(\bar{\theta}, \bar{f}\right)$ *has a singularity if and only if* $\left(\bar{\theta}, \bar{f}\right) \in S$.

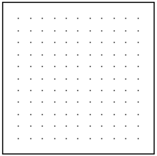
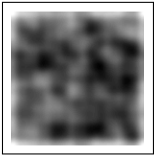
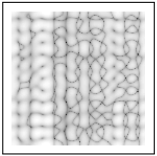
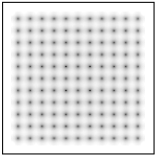
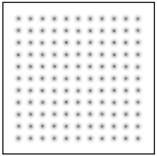
Intuition:

- Coprime arrays identify up to $M_1 N_1$ DOAs.
- Coprime samplers identify up to $M_2 N_2$ frequencies.

[9] P. Pal and P. P. Vaidyanathan, "Nested arrays: a novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, 2010.

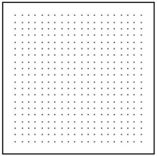
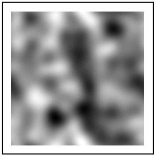
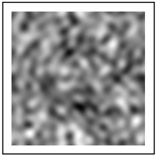
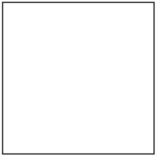
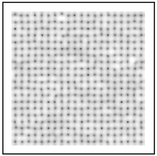
Outline

Angle-Doppler patterns: 121 Sources

Ideal	PSD	MVDR	ULA MUSIC	coprime JADE
				

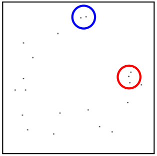
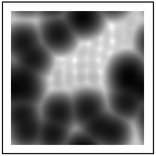
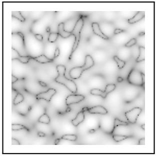
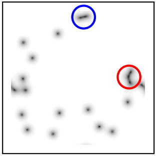
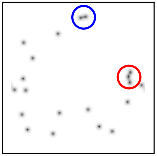
- 11×11 uniformly-placed sources, 1000 snapshots and 0dB SNR.
- $M_1 = M_2 = 4$ and $N_1 = N_2 = 5$ so that the number of sensors and the number of samples per sensor are 12 in all methods.
- Performance limit of ULA $+$ uniform sampling.

Angle-Doppler patterns: 400 Sources

Ideal	PSD	MVDR	ULA MUSIC	coprime JADE
				

- 20×20 uniformly-placed sources, 1000 snapshots and 0dB SNR.
- $M_1 = M_2 = 4$ and $N_1 = N_2 = 5$ so that the number of sensors and the number of samples per sensor are 12 in all methods.
- Performance limit of coprime array $+$ coprime sampling.

Angle-Doppler patterns: 20 Sources

Ideal	PSD	MVDR	ULA MUSIC	coprime JADE
				

- 20 randomly-placed sources, 1000 snapshots and 0dB SNR.
- $M_1 = M_2 = 4$ and $N_1 = N_2 = 5$ so that the number of sensors and the number of samples per sensor are 12 in all methods.
- Coprime JADE has the best performance in terms of spread-out.

Outline

Conclusion

- Coprime JADE = Coprime arrays + Coprime samplers.
- Coprime JADE identifies up to $O(M_1 N_1 M_2 N_2)$ sources.
    - STAP only achieves $O((M_1 + N_1)(M_2 + N_2))$ degrees of freedom.
- Future work:
    - Connection between coprime JADE and coprime arrays in high dimensions[10].
    - Different coprime sampling schemes at different sensors.

Thank you for your attention!

[10] P. P. Vaidyanathan and P. Pal, "Theory of sparse coprime sensing in multiple dimensions," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3592–3608, 2011.