Design of Coprime DFT Arrays and Filter Banks

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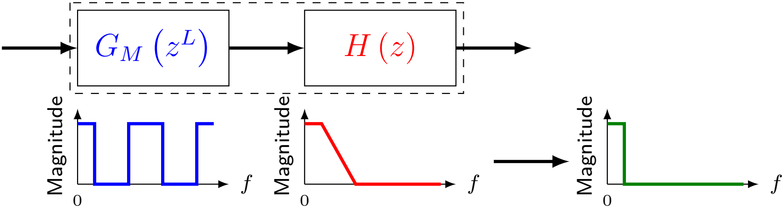
Motivation

- Coprime DFT filter banks: [1]
  - Enhanced degrees of freedom: $O(MN)$ based on $O(M+N)$ samples.
  - Applications in Direction-of-arrival estimation [2], Beamforming [3], and Spectrum estimation [4].
  - Problems: No design guidelines!
- Interpolated FIR filter design [5]–[7]: IFIR design $\rightarrow$ two filter designs.

$$F_i(z) = G_M\left(z^L\right) H(z)$$



Coprime DFT Filter Banks

$G(z), H(z)$:
FIR Type-I filters
N_g, N_h: filter orders

$$G(z) = \sum_{n=0}^{N_g} g(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{N_h} h(n) z^{-n}$$

↓

$G(z^M), H(z^N)$:
sparse coefficient filters

$$G\left(z^M\right) \qquad H\left(z^N\right)$$

↓

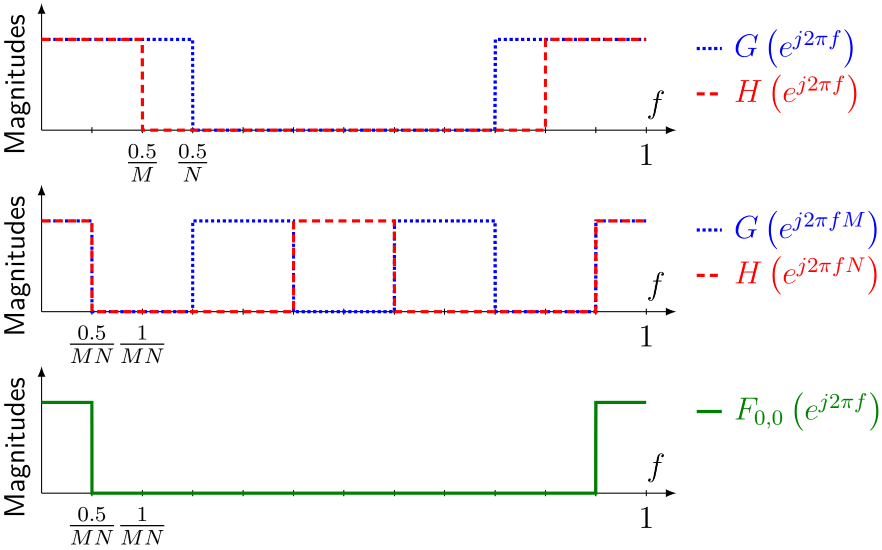
$$F_{\ell,k}(z) \leftarrow G\left(z^M W_N^{\ell}\right) \times H\left(z^N W_M^{k}\right)$$

coprime DFT filter banks
MN-filters
$\ell = 0, 1, \ldots, N-1,$
$k = 0, 1, \ldots, M-1.$

DFT filter banks
N-filters
$\ell = 0, 1, \ldots, N-1.$

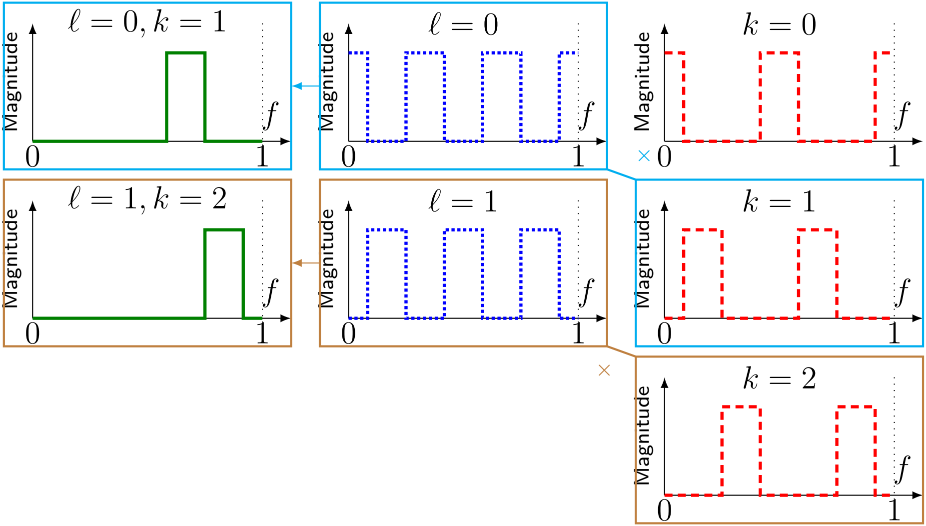
DFT filter banks
M-filters
$k = 0, 1, \ldots, M-1.$

Coprime DFTFBs, the Ideal Case

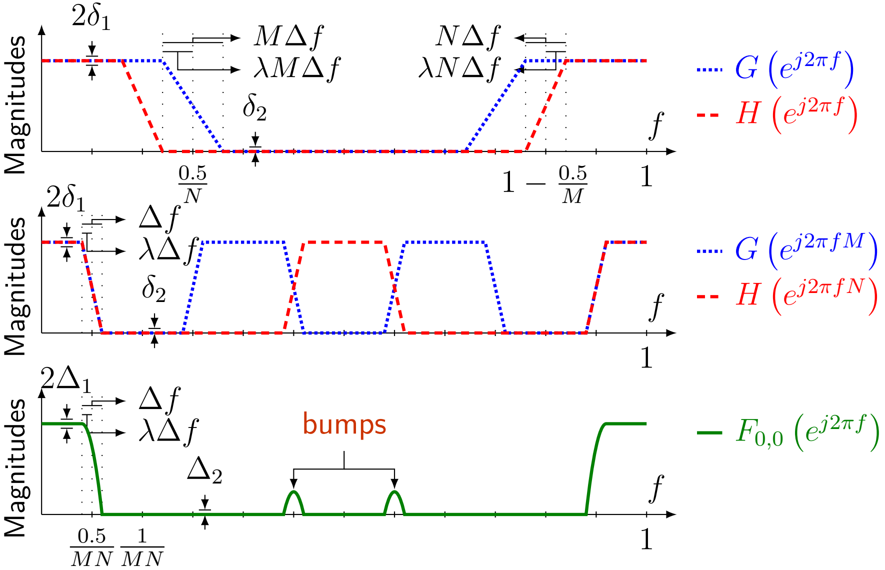


Coprime DFTFBs, the Ideal Case

$$F_{\ell,k}\left(e^{j2\pi f}\right) = G\left(e^{j2\pi f M} e^{-j2\pi \ell/N}\right) \times H\left(e^{j2\pi f N} e^{-j2\pi k/M}\right)$$

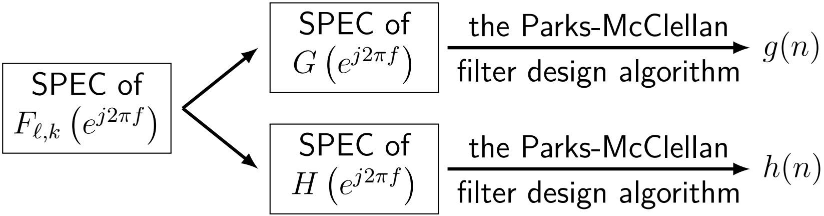


Coprime DFTFBs, the Practical Case



Coprime DFTFBs, Design Parameters

- Goal: Design $g(n)$ and $h(n)$ such that $F_{\ell,k}\left(e^{j2\pi f}\right)$ is an approximation of the ideal case.
- Notion of *approximation* in $F_{\ell,k}\left(e^{j2\pi f}\right)$:
  1. Passband ripples Δ_1,
  2. Stopband ripples Δ_2,
  3. Transition band width Δf,
  4. Passband edges and stopband edges.
- Idea: Divide the design problem into two sub-problems.



Design equations

- Passband ripples and stopband ripples for $G\left(e^{j2\pi f}\right)$ and $H\left(e^{j2\pi f}\right)$,

$$\delta_1 = 1 - \sqrt{1-\Delta_1}, \qquad \delta_2 = \frac{\Delta_2}{2-\sqrt{1-\Delta_1}}.$$

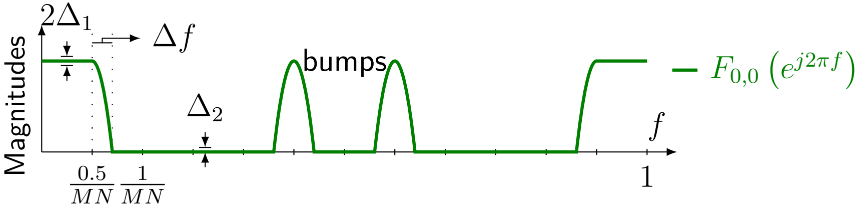
- Transition bandwidth Δf,

$$\Delta f \geq \frac{2\log_{10}\left(\frac{1}{10\delta_1\delta_2}\right)}{3\min\left\{MN_g, NN_h\right\}}.$$

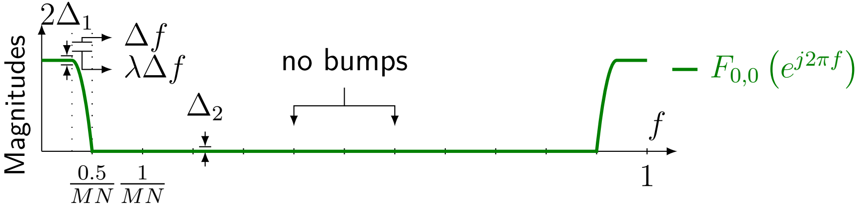
- λ determines passband edges and stopband edges.

The Role of λ

- $\lambda = 0$, larger passband edges ☺, with bumps ☹.



- $\lambda = 1$, smaller passband edges ☹, no bumps ☺.



Optimal λ

- Find the one with maximal passband edge subject to "bump-free" constraints.

$$\lambda_{opt} = \min_{\lambda} \ \lambda \quad \text{subject to} \quad \left| F_{0,0}\left(e^{j2\pi f}\right) \right| \leq \Delta_2,$$
$$f \in \left[\frac{0.5}{MN} + (1-\lambda)\Delta f, 1 - \frac{0.5}{MN} - (1-\lambda)\Delta f \right],$$

- Relaxation

$$\hat{\lambda} = \min_{\lambda} \ \lambda \quad \text{subject to} \quad \left| \hat{F}_{00}\left(e^{j2\pi f}\right) \right| \leq \Delta_2,$$
$$f \in \left[\frac{0.5}{MN} + (1-\lambda)\Delta f, 1 - \frac{0.5}{MN} - (1-\lambda)\Delta f \right],$$

- $\hat{F}_{00}\left(e^{j2\pi f}\right)$ might not be realizable in the FIR setting but can be written as some simple closed-form functions.

Approximate the transition bands

- Substitute transition bands of $G\left(e^{j2\pi f}\right)$ and $H\left(e^{j2\pi f}\right)$ with simple closed-form functions.
- Linear functions,

$$\lambda \geq \hat{\lambda}_{li} \triangleq \boxed{\frac{1-\delta_1-\sqrt{\Delta_2}}{1-\delta_1-\delta_2}}.$$

- Q functions,

$$\lambda \geq \hat{\lambda}_{Q} \triangleq \boxed{\frac{Q_1-\sqrt{-\ln\left(4\Delta_2\right)}}{Q_1-Q_2}},$$

  - $Q_1 \triangleq Q^{-1}\left(1-\delta_1\right)$,
  - $Q_2 \triangleq Q^{-1}\left(\delta_2\right)$,
  - $Q^{-1}\left(\ \cdot\ \right)$: inverse Q functions.

Approximate the transition bands

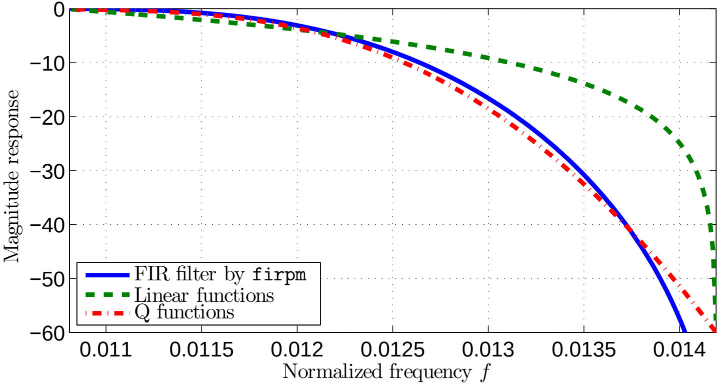


Figure : A comparison among different approximations of the transition band. The FIR filter is designed by the `MATLAB` function `firpm` with specification $f_p = 0.0108$, $f_s = 0.0142$, $\delta_1 = 0.01$, and $\delta_2 = 0.001$. The filter order is 160.

Summary: Coprime DFTFB Design

Inputs: $(M, N, N_g, N_h, \Delta_1, \Delta_2)$

Initialize:

$\delta_1 = 1 - \sqrt{1-\Delta_1}, \qquad \delta_2 = \Delta_2/(2 - \sqrt{1-\Delta_1}),$

$\Delta f \geq 2\log_1 0\left(\frac{1}{10\delta_1\delta_2}\right)/(2\min\{MN_g, NN_h\}),$

$\lambda = \hat{\lambda}_{li} = (1 - \delta_1 - \sqrt{\Delta_2})/(1-\delta_1-\delta_2)$

or $\hat{\lambda}_Q = (Q_1 - \sqrt{-\ln(4\Delta_2)})/(Q_1 - Q_2),$

Increase λ until stopband ripples for $F_{0,0}\left(e^{j2\pi f}\right)$ are satisfied.

Increase Δf until ripples for $G\left(e^{j2\pi f}\right)$ and $H\left(e^{j2\pi f}\right)$ are met.

Design lowpass filters $g(n)$ and $h(n)$ with specifications

$(\delta_1, \delta_2, 0.5/N - \lambda M\Delta f, 0.5/N + (1-\lambda)M\Delta f)$ for $g(n)$,

$(\delta_1, \delta_2, 0.5/M - \lambda N\Delta f, 0.5/M + (1-\lambda)N\Delta f)$ for $h(n)$.

Output: $(g(n), h(n))$

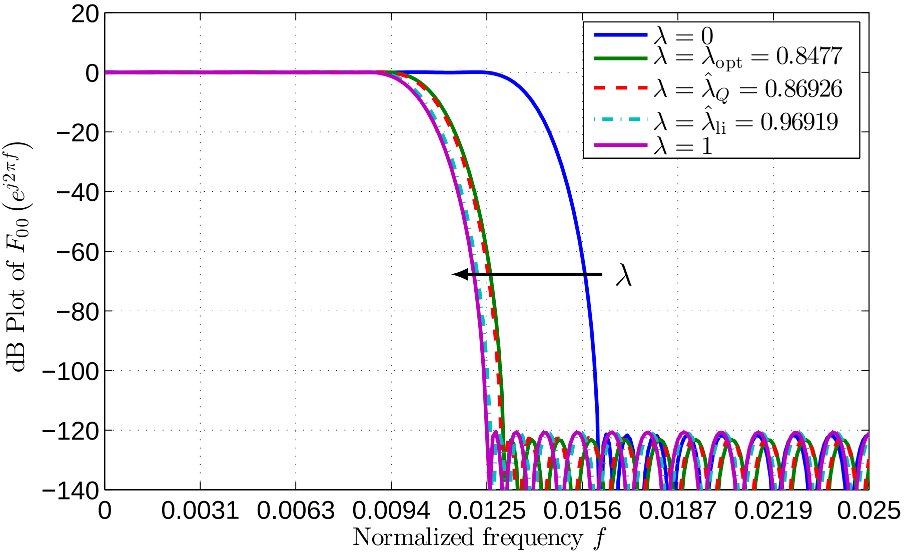
Numerical Example

- $M = 8, N = 5,$
- $N_g = 100, N_h = 160,$
- $\Delta_1 = 0.01, \Delta_2 = 0.001.$
- Define overall amplitude response $A\left(e^{j2\pi f}\right)$,

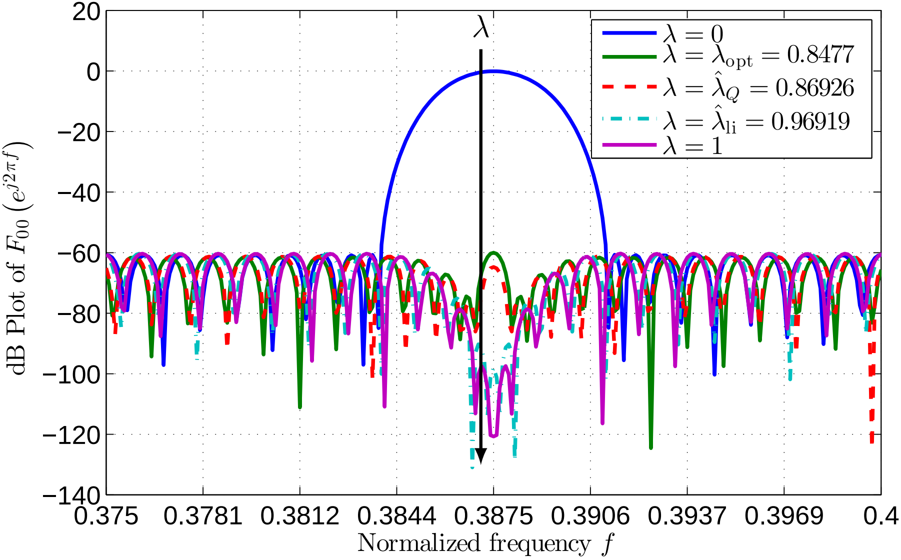
$$A\left(e^{j2\pi f}\right) = \sum_{\ell=0}^{N-1} \sum_{k=0}^{M-1} \left|F_{\ell,k}\left(e^{j2\pi f}\right)\right|,$$

to measure the *spectral coverage*.

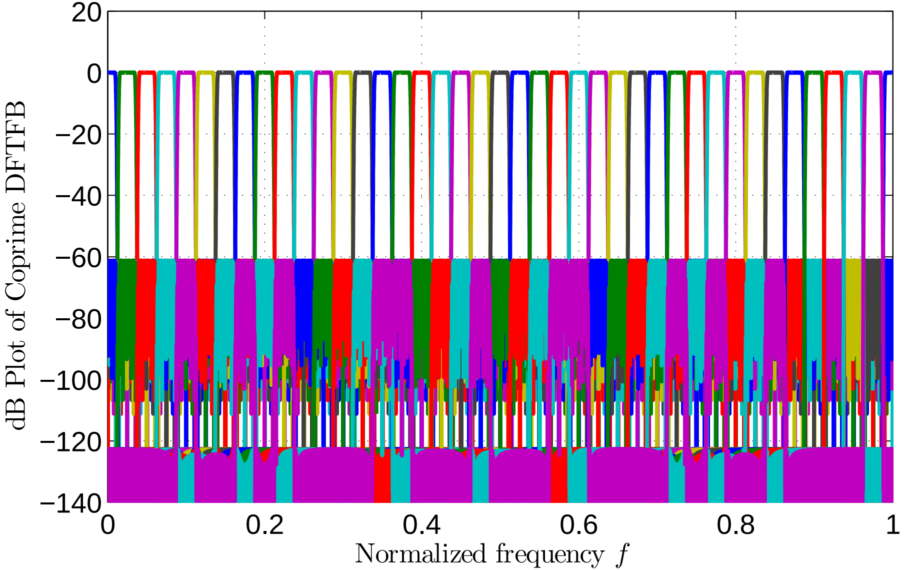
Passband Characteristics



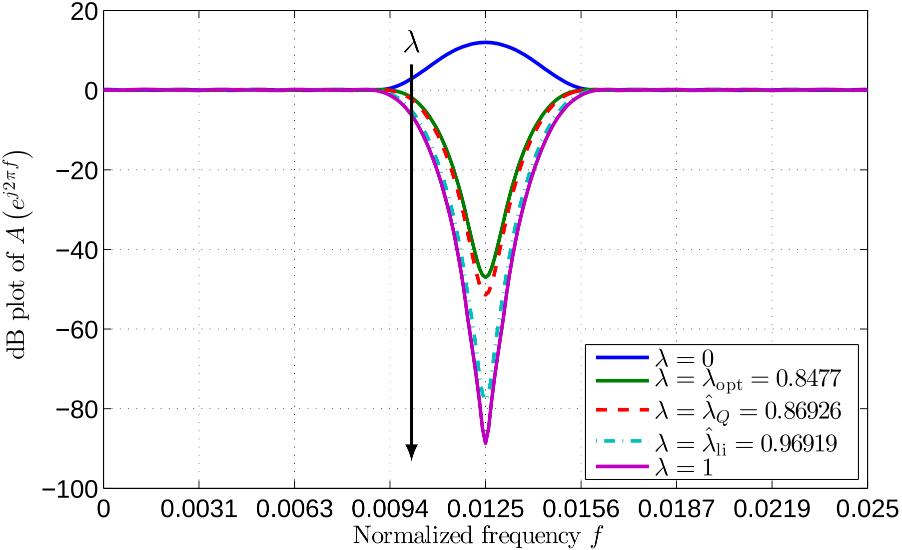
Stopband Characteristics



Spectrum Coverage



Overall Amplitude Responses



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