The Role of Difference Coarrays in Correlation Subspaces

Chun-Lin Liu and P. P. Vaidyanathan
Dept. of Electrical Engineering, 136-93
California Institute of Technology, Pasadena, CA 91125, USA
cl.liu@caltech.edu, ppvnath@systems.caltech.edu

***Abstract*—The concept of correlation subspaces was recently introduced in array processing literature by Rahmani and Atia. Given a sensor array, its geometry determines the correlation subspace completely, and the covariance matrix of the array output is constrained in a certain way by the correlation subspace. It has been shown by Rahmani and Atia that this knowledge about the covariance constraint can be exploited to improve the performance of DOA estimators. In this paper, it is shown that there is a simple closed form expression for the basis vectors of the correlation subspace. Thus, computation of this subspace is greatly simplified. Another fundamental observation is that, this expression is closely related to the difference coarray. Thirdly, the paper also shows an interesting logical connection between correlation subspaces, redundancy averaging, and rectification, which are popularly used in DOA estimation[1].**

***Index Terms*—Correlation subspace, difference coarray, redundancy averaging, DOA estimation.**

I. Introduction

Direction-of-arrival (DOA) estimation has been a popular research field in array processing for several decades, which finds useful applications in radio astronomy, radar, imaging, and communications [1]–[3]. DOA estimators such as MUSIC [4], ESPRIT [5], and SPICE [6], to name a few [7]–[11] have been developed for these applications.

Recently, the elegant concept of *correlation subspaces* was proposed by Rahmani and Atia [12], to improve DOA estimation in a number of ways. For any array geometry, the correlation subspace is uniquely determined, and imposes some implicit constraints on the structure of the covariance, as we shall see. This subspace can be utilized in the denoised covariance matrix, from which the source directions are estimated more accurately [12]. Note that the correlation subspace depends on the array configurations and prior knowledge about the sources but is independent of the choice of DOA estimators. Hence, a broad class of DOA estimators are applicable to the denoised covariance matrix. However, the explicit expressions for the correlation subspace were not known, so its approximation was computed numerically in [12]. Furthermore, the way in which the correlation subspace is influenced by the array configuration was not explored.

Inspired by correlation subspaces introduced in [12], this paper makes a number of new contributions. We first generalize the result in [12] to formulate what we call the *generalized correlation subspace*. This makes it possible to analyze the correlation subspace readily. Furthermore, we will show how to obtain simple and elegant closed-form expressions for the dimension and the basis vectors of the correlation subspace, in terms of the sensor geometry and the *difference coarray* geometry. Note that the difference coarray [13] was previously found to be important in the study of sparse arrays [8], [9]. The results of this paper not only simply the computation of the correlation subspace significantly, but also provide insights into some well-known array processing techniques, such as redundancy averaging [7] and rectification [14].

The outline of this paper is as follows: Section II reviews the correlation subspace. Section III first proposes the generalized correlation subspace and then derives the closed-form expressions for the correlation subspace. Section IV discusses the connections with redundancy averaging and rectification. Section V presents demonstrative examples while Section VI concludes this paper.

Notations: Scalars, vectors, matrices, and sets are denoted by lowercase letters (a), lowercase letters in boldface ($\mathbf{a}$), uppercase letters in boldface ($\mathbf{A}$), and letters in blackboard boldface ($\mathbb{A}$), respectively. $\mathbf{A}^T$, $\mathbf{A}^*$, and $\mathbf{A}^H$ are the transpose, complex conjugate, and complex conjugate transpose of $\mathbf{A}$. The Moore-Penrose pseudoinverse of $\mathbf{A}$ is $\mathbf{A}^\dagger$. If $\mathbf{A}$ has full column rank, then $\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$. The vectorization operator is defined as $\mathrm{vec}([\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N]) = [\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_N^T]^T$, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ are column vectors. For two Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \succeq \mathbf{B}$ is equivalent to $\mathbf{A} - \mathbf{B}$ being positive semidefinite. $\mathrm{col}(\mathbf{A})$ stands for the column space of $\mathbf{A}$. The support of a function $f(x)$ is defined as $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$, where x belongs to the domain of f. The indicator function $\mathbf{1}_{\mathbb{A}}(x)$ is one if $x \in \mathbb{A}$ and zero otherwise. $\mathbb{E}[\cdot]$ is the expectation operator. The cardinality of a set $\mathbb{A}$ is denoted by $|\mathbb{A}|$.

The bracket notation [15], [16] is reviewed as follows. Assume the sensor locations are characterized by a set $\mathbb{S} = \{0, 3, 6\}$. Assume the array output on $\mathbb{S}$ is denoted by $\mathbf{x}_{\mathbb{S}} = [7, 8, 9]^T$. The square bracket $[\mathbf{x}_{\mathbb{S}}]_i$ represents the ith entry of $\mathbf{x}_{\mathbb{S}}$ while the triangular bracket $\langle \mathbf{x}_{\mathbb{S}} \rangle_n$ denotes the sample value on the support location n. Hence, we have $[\mathbf{x}_{\mathbb{S}}]_1 = 7$, $[\mathbf{x}_{\mathbb{S}}]_2 = 8$, $[\mathbf{x}_{\mathbb{S}}]_3 = 9$, $\langle \mathbf{x}_{\mathbb{S}} \rangle_0 = 7$, $\langle \mathbf{x}_{\mathbb{S}} \rangle_3 = 8$, and $\langle \mathbf{x}_{\mathbb{S}} \rangle_6 = 9$. Similar notations apply to matrices. For instance, if $\mathbf{A} = \mathbf{x}_{\mathbb{S}}\mathbf{x}_{\mathbb{S}}^T$, then $[\mathbf{A}]_{i,j} = [\mathbf{x}_{\mathbb{S}}]_i[\mathbf{x}_{\mathbb{S}}]_j$ and $\langle \mathbf{A} \rangle_{n_1,n_2} = \langle \mathbf{x}_{\mathbb{S}} \rangle_{n_1} \langle \mathbf{x}_{\mathbb{S}} \rangle_{n_2}$.

II. Review of Correlation Subspaces

Assume that D monochromatic sources impinge on an one-dimensional sensor array. The sensor locations are $n\lambda/2$, where n belongs to an integer set $\mathbb{S} \subset \mathbb{Z}$ and λ is the wavelength. Let $\theta_i \in [-\pi/2, \pi/2]$ be the DOA of the ith source. The normalized DOA of the ith source is defined as $\bar{\theta}_i = (\sin\theta_i)/2 \in [-1/2, 1/2]$. The measurements on the

[1]This work was supported in parts by the ONR grants N00014-15-1-2118 and N00014-17-1-2732, the NSF grant CCF-1712633, and the California Institute of Technology.

sensor array $\mathbb{S}$ can be modeled as

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathbb{S}} \quad \in \mathbb{C}^{|\mathbb{S}|}, \tag{1}$$

where A_i is the complex amplitude of the ith source, $\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) = [e^{j2\pi\bar{\theta}_i n}]_{n\in\mathbb{S}}$ are the steering vectors, and $\mathbf{n}_{\mathbb{S}}$ is the additive noise term. It is assumed that the sources and noise are zero-mean and *uncorrelated*. Namely, let $\mathbf{s} = [A_1, \ldots, A_D, \mathbf{n}_{\mathbb{S}}^T]^T$. Then $\mathbb{E}[\mathbf{s}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathrm{diag}(p_1, \ldots, p_D, p_n, \ldots, p_n)$, where p_i and p_n are the power of the ith sources and the noise, respectively.

The covariance matrix of $\mathbf{x}_{\mathbb{S}}$ can be expressed as

$$\mathbf{R}_{\mathbb{S}} = \mathbb{E}[\mathbf{x}_{\mathbb{S}}\mathbf{x}_{\mathbb{S}}^H] = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + p_n\mathbf{I}. \tag{2}$$

Rearranging the elements in (2) leads to

$$\mathrm{vec}(\mathbf{R}_{\mathbb{S}} - p_n\mathbf{I}) = \sum_{i=1}^{D} p_i \mathbf{c}(\bar{\theta}_i), \tag{3}$$

where the correlation vectors $\mathbf{c}(\bar{\theta}_i)$ are defined as

$$\mathbf{c}(\bar{\theta}_i) \triangleq \mathrm{vec}(\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i)) \quad \in \mathbb{C}^{|\mathbb{S}|^2}. \tag{4}$$

The relation (3) implies

$$\begin{aligned}\mathrm{vec}(\mathbf{R}_{\mathbb{S}} - p_n\mathbf{I}) &\in \mathrm{span}\{\mathbf{c}(\bar{\theta}_i) : i = 1, 2, \ldots, D\} &(5)\\ &\subseteq \mathcal{CS} \triangleq \mathrm{span}\{\mathbf{c}(\bar{\theta}) : -1/2 \leq \bar{\theta} \leq 1/2\}, &(6)\end{aligned}$$

where the linear span in (6) is defined as the set of *all* vectors of the form $\sum_{\ell=1}^{L} a_\ell \mathbf{c}(\bar{\theta}_\ell)$ where $L \in \mathbb{N}$, $a_\ell \in \mathbb{C}$, and $-1/2 \leq \bar{\theta}_\ell \leq 1/2$ [17]. This subspace is called the *correlation subspace*, denoted by $\mathcal{CS}$. Eq. (6) also indicates that $\mathrm{vec}(\mathbf{R}_{\mathbb{S}} - p_n\mathbf{I})$ is constrained in a certain way by $\mathcal{CS}$, and these constraints can be used in designing DOA estimators for improved performance.

It is clear that $\mathcal{CS}$ is a finite-dimensional subspace of $\mathbb{C}^{|\mathbb{S}|^2}$, due to (4). However, the definition of correlation subspace in (6) is computationally intractable since it involves infinitely many $\mathbf{c}(\bar{\theta})$. Alternatively, It was observed in [12] that the correlation subspace can be computed through the column space of a matrix of finite size, as follows:

Observation 1. *The correlation subspace* $\mathcal{CS}$ *satisfies*

$$\mathcal{CS} = \mathrm{col}(\mathbf{S}), \tag{7}$$

where the correlation subspace matrix $\mathbf{S}$ *is defined as*

$$\mathbf{S} \triangleq \int_{-\pi/2}^{\pi/2} \mathbf{c}(\bar{\theta})\mathbf{c}^H(\bar{\theta})\mathrm{d}\theta \quad \in \mathbb{C}^{|\mathbb{S}|^2 \times |\mathbb{S}|^2}. \tag{8}$$

The proof for this observation is given in [18]. Note that this integral is carried out over the DOA, $\theta \in [-\pi/2, \pi/2]$ and the relation $\bar{\theta} = (\sin\theta)/2$ can be utilized to evaluate (8). According to (8), it can be shown that the correlation subspace matrix $\mathbf{S}$ is Hermitian and positive semidefinite.

It was shown in [12] that, the right-hand side of (7) can be simplified further, based on the eigenvectors of $\mathbf{S}$ associated with the nonzero eigenvalues. In particular, let the eigen-decomposition of $\mathbf{S}$ be

$$\mathbf{S} = \underbrace{\begin{bmatrix}\mathbf{Q}_{\mathcal{CS}} & \mathbf{Q}_{\mathcal{CS}^\perp}\end{bmatrix}}_{\mathbf{Q}} \begin{bmatrix}\mathbf{\Lambda}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{Q}_{\mathcal{CS}} & \mathbf{Q}_{\mathcal{CS}^\perp}\end{bmatrix}^H, \tag{9}$$

where the diagonal matrix $\mathbf{\Lambda}_1$ contains the positive eigenvalues in the descending order and the columns of $\mathbf{Q}$ consist of the orthonormal eigenvectors. Then, (7) and (9) lead to $\mathcal{CS} = \mathrm{col}(\mathbf{Q}_{\mathcal{CS}})$. Namely, the correlation subspace $\mathcal{CS}$ is the column space of the matrix $\mathbf{Q}_{\mathcal{CS}}$. Eqs. (7), (8), and (9) indicate that the matrix $\mathbf{S}$, its eigenvalues, its eigenvectors, and the correlation subspace *depend purely on the array configuration*.

For any array geometry, the correlation subspace is uniquely determined, and imposes some implicit constraints on the structure of the covariance matrix, as indicated in (6). This admits a covariance-matrix denoising approach [12]. To begin with, consider finite snapshot sensor measurements $\widetilde{\mathbf{x}}_{\mathbb{S}}(k)$ for $k = 1, \ldots, K$. The sample covariance matrix $\widetilde{\mathbf{R}}_{\mathbb{S}}$ becomes

$$\widetilde{\mathbf{R}}_{\mathbb{S}} = \frac{1}{K}\sum_{k=1}^{K} \widetilde{\mathbf{x}}_{\mathbb{S}}(k)\widetilde{\mathbf{x}}_{\mathbb{S}}^H(k). \tag{10}$$

The algorithm in [12] first denoises the sample covariance matrix $\widetilde{\mathbf{R}}_{\mathbb{S}}$ using the following convex program (P1):

$$\begin{aligned}\text{(P1): } \mathbf{R}_{\text{P1}}^\star \triangleq \arg\min_{\mathbf{R}} \ & \|\widetilde{\mathbf{R}}_{\mathbb{S}} - p_n\mathbf{I} - \mathbf{R}\|_2^2 &(11)\\ \text{subject to } & (\mathbf{I} - \mathbf{Q}_{\mathcal{CS}}\mathbf{Q}_{\mathcal{CS}}^\dagger)\mathrm{vec}(\mathbf{R}) = \mathbf{0}, &(12)\\ & \mathbf{R} \succeq \mathbf{0}, &(13)\end{aligned}$$

where the noise power p_n is estimated from the eigenvalues of $\widetilde{\mathbf{R}}_{\mathbb{S}}$ and $\|\cdot\|_2$ denotes the spectral norm of a matrix (i.e., the largest singular value). The cost function in (11) suggests that the matrix $\mathbf{R}_{\text{P1}}^\star$ resembles $\sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i)$ in (2). The constraint (12) ensures that $\mathrm{vec}(\mathbf{R}_{\text{P1}}^\star)$ belongs to the correlation subspace while (13) indicates that $\mathbf{R}_{\text{P1}}^\star$ is positive semidefinite. Furthermore, (P1) has to be solved numerically, due to lack of closed-form solutions [12].

Note that the solution $\mathbf{R}_{\text{P1}}^\star$ can be used in a broad class of state-of-the-art DOA estimators, such as MUSIC [4], ESPRIT [5], SPICE [6], and coarray MUSIC [7], [8]. This approach leads to better estimation performance than DOA estimators on $\widetilde{\mathbf{R}}_{\mathbb{S}}$, as demonstrated in [12] using the MUSIC algorithm.

To implement problem (P1), it is crucial to find the matrix $\mathbf{Q}_{\mathcal{CS}}$ first. This step needs to be done only once per array. Once the matrix $\mathbf{Q}_{\mathcal{CS}}$ is obtained, as in (9), it can be used repeatedly in problem (P1). To calculate $\mathbf{Q}_{\mathcal{CS}}$, the numerical integration was utilized in [12]. This step is typically done by choosing a dense grid of the parameter θ, which only *approximates* the integral in (8). Furthermore, the *numerical* eigen-decomposition in (9) introduces perturbations on zero eigenvalues, making it challenging to determine the correlation subspace precisely. It is desirable to mitigate these negative effects caused by numerical computations, as we do next.

III. Main Results

The main difficulty in deriving the closed-form expressions for $\mathbf{Q}_{\mathcal{CS}}$ is as follows. It can be shown that the entries of $\mathbf{S}$ are related to Bessel functions, making it complicated to obtain analytical forms of (9). In this section, we will present generalized correlation subspaces, which enable us to derive explicit expressions for the correlation subspace.

A. Generalized Correlation Subspaces

As a motivating example, let us consider the definition of $\mathbf{S}$ in (8). Since $\bar{\theta} = (\sin\theta)/2$, we have $\mathrm{d}\theta = 2(1-(2\bar{\theta})^2)^{-1/2}\mathrm{d}\bar{\theta}$. Hence, (8) can be rewritten as

$$\mathbf{S} = \int_{-1/2}^{1/2} \mathbf{c}(\bar{\theta})\mathbf{c}^H(\bar{\theta}) \underbrace{\left(2(1-(2\bar{\theta})^2)^{-1/2}\right)}_{\text{the density function}} \mathrm{d}\bar{\theta}. \tag{14}$$

Note that (14) can be regarded as a weighted integral with the density function $2(1-(2\bar{\theta})^2)^{-1/2}$ over $\bar{\theta} \in [-1/2, 1/2]$. Hence, we can generalize the correlation subspace matrix by varying the density function in (14). It is formally defined as

Definition 1. *Let the correlation vector $\mathbf{c}(\bar{\theta})$ be defined as in* (4)*. Let $\rho(\bar{\theta})$ be a nonnegative Lebesgue integrable function over the set* $[-1/2, 1/2]$*. The generalized correlation subspace matrix associated with $\rho(\bar{\theta})$ is defined as*

$$\mathbf{S}(\rho) \triangleq \int_{-1/2}^{1/2} \mathbf{c}(\bar{\theta})\mathbf{c}^H(\bar{\theta})\rho(\bar{\theta})\mathrm{d}\bar{\theta}. \tag{15}$$

It can be seen that (14) is a special case of Definition 1, with $\rho(\bar{\theta}) = 2(1-(2\bar{\theta})^2)^{-1/2}\mathbf{1}_{[-1/2,1/2]}(\bar{\theta})$. The density function $\rho(\bar{\theta})$ quantifies the importance of $\mathbf{c}(\bar{\theta})\mathbf{c}^H(\bar{\theta})$ in $\mathbf{S}(\rho)$ across different $\bar{\theta}$. Then the generalized correlation subspace can be defined as follows:

Definition 2. *Let $\mathbf{S}(\rho)$ be the generalized correlation subspace matrix associated with $\rho(\bar{\theta})$, as in* (15)*. The generalized correlation subspace is defined as $\mathcal{GCS}(\rho) \triangleq \mathrm{col}(\mathbf{S}(\rho))$.*

It can be seen from Definition 1 and 2 that the generalized correlation subspaces are parameterized by the density function $\rho(\bar{\theta})$. For any given support of $\rho(\bar{\theta})$, the generalized correlation subspace is invariant to the exact shape of $\rho(\bar{\theta})$ under that support, as indicated by the following lemma [18]:

Lemma 1. *Let $\rho_1(\bar{\theta})$ and $\rho_2(\bar{\theta})$ be two nonnegative Lebesgue integrable functions over the set $[-1/2, 1/2]$. If $\mathrm{supp}(\rho_1) = \mathrm{supp}(\rho_2)$, then $\mathcal{GCS}(\rho_1) = \mathcal{GCS}(\rho_2)$.*

Corollary 1. *Let the density function in* (14) *be $\rho_1(\bar{\theta}) = 2(1-(2\bar{\theta})^2)^{-1/2}\mathbf{1}_{[-1/2,1/2]}(\bar{\theta})$ and the constant density function be $\rho_2(\bar{\theta}) = \mathbf{1}_{[-1/2,1/2]}(\bar{\theta})$. Then $\mathcal{CS} = \mathcal{GCS}(\rho_1) = \mathcal{GCS}(\rho_2)$.*

Corollary 1 also enables us to analyze the correlation subspace readily through the generalized correlation subspace $\mathcal{GCS}(\rho_2)$, which will be developed in Section III-B.

B. Closed-Form Expressions of Correlation Subspace

In this section, the closed-form expressions of the correlation subspace will be investigated. This will reveal a fundamental connection between the correlation subspace and the difference coarray. Before presenting these results, first we need to define the difference coarray, the matrix $\mathbf{J}$, and the weight function for any array geometry $\mathbb{S}$ as follows [16]:

Definition 3 (Difference coarray)**.** *The difference coarray $\mathbb{D}$ contains the differences between the elements in $\mathbb{S}$, i.e., $\mathbb{D} \triangleq \{n_1 - n_2 : \forall\, n_1, n_2 \in \mathbb{S}\}$.*

Definition 4 (The matrix $\mathbf{J}$)**.** *The binary matrix $\mathbf{J}$ has size $|\mathbb{S}|^2$-by-$|\mathbb{D}|$. The columns of $\mathbf{J}$ satisfy $\langle \mathbf{J} \rangle_{:,m} = \mathrm{vec}(\mathbf{I}(m))$ for $m \in \mathbb{D}$, where $\mathbf{I}(m) \in \{0,1\}^{|\mathbb{S}|\times|\mathbb{S}|}$ is given by*

$$\langle \mathbf{I}(m) \rangle_{n_1,n_2} = \begin{cases} 1, & \text{if } n_1 - n_2 = m, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Here the bracket notation $\langle \cdot \rangle_{n_1,n_2}$ is defined in Section I.

Definition 5 (Weight function $w(m)$)**.** *The weight function $w(m)$ of an array $\mathbb{S}$ is defined as the number of sensor pairs with coarray index m. Namely, $w(m) \triangleq |\{(n_1,n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\}|$ for $m \in \mathbb{D}$.*

In addition, it can be shown that the matrix $\mathbf{J}$ has orthogonal columns, as in the following lemma [16], [18]:

Lemma 2. $\mathbf{J}^H\mathbf{J} = \mathbf{W} \triangleq \mathrm{diag}(w(m))_{m\in\mathbb{D}}$*. Namely, $\mathbf{J}$ has orthogonal columns and the norm of the column associated with the coarray index m is $\sqrt{w(m)}$.*

Now we proceed to derive the closed forms for the correlation subspace. Using the above results and Definition 1 with the density function $\rho_2(\bar{\theta}) = \mathbf{1}_{[-1/2,1/2]}(\bar{\theta})$, we have the following lemma [18]:

Lemma 3. *The eigen-decomposition of $\mathbf{S}(\rho_2)$ is given by*

$$\mathbf{S}(\rho_2) = (\mathbf{J}\mathbf{W}^{-1/2})\mathbf{W}(\mathbf{J}\mathbf{W}^{-1/2})^H, \tag{17}$$

where the matrix $\mathbf{J}$ and $\mathbf{W}$ are defined in Definition 4 and Lemma 2, respectively.

Eq. (17) also indicates that the matrix $\mathbf{J}\mathbf{W}^{-1/2}$ corresponds to the orthonormal eigenvectors while the diagonal matrix $\mathbf{W}$ is associated with the eigenvalues. In particular, the *positive eigenvalues* and the associated eigenvectors of $\mathbf{S}(\rho_2)$ are given by

$$\text{Positive eigenvalues of } \mathbf{S}(\rho_2) = w(m), \tag{18}$$

$$\text{Eigenvectors of } \mathbf{S}(\rho_2) = \frac{\mathrm{vec}(\mathbf{I}(m))}{\sqrt{w(m)}}, \tag{19}$$

where $m \in \mathbb{D}$. Note that (18) and (19) can be calculated readily from the array geometry using Definition 5 and 4, respectively. Namely, the eigen-decomposition of $\mathbf{S}(\rho_2)$ can be evaluated without using the numerical integration in Definition 1 and the numerical eigen-decomposition on $\mathbf{S}(\rho_2)$.

Theorem 1. *Let the matrix $\mathbf{J}$ be defined in Definition 4. Then the correlation subspace satisfies*

$$\mathcal{CS} = \mathrm{col}(\mathbf{J}). \tag{20}$$

Proof: According to Corollary 1, we obtain $\mathcal{CS} = \mathcal{GCS}(\rho_1) = \mathcal{GCS}(\rho_2)$. The relation $\mathcal{GCS}(\rho_2) = \mathrm{col}(\mathbf{J})$ is due to Definition 2 and Lemma 3. ∎

This theorem indicates that the correlation subspace is fully characterized by the matrix $\mathbf{J}$, which can be readily computed from *sensor locations and the difference coarray*. Namely, to compute the correlation subspace, the numerical integration (8) and the eigen-decomposition (9) can be avoided completely. Due to Theorem 1 and Lemma 2, the dimension of the correlation subspace is given by

Corollary 2. *The dimension of the correlation subspace is the size of the difference coarray, i.e., $\dim(\mathcal{CS}) = |\mathbb{D}|$.*

IV. Connections with Redundancy Averaging and Rectification

In this section, we will discuss a covariance matrix denoising framework based on orthogonal projections onto the correlation subspace. This method, denoted by problem (P2), can be regarded as a modified version of the optimization problem (P1). This problem (P2) can be solved by simple, closed-form, and cost-effective expressions, unlike the problem (P1). Furthermore, (P2) is closely related to *redundancy averaging*, which is a well-known processing technique in coarray-based DOA estimators.

The rationale for (P2) is based on the following chain of arguments. By setting $m = 0$ in (16), it can be shown that $p_n \text{vec}(\mathbf{I}) \in \text{col}(\mathbf{J})$, where $\mathbf{I}$ is the identity matrix. This means $p_n \text{vec}(\mathbf{I}) \in \mathcal{CS}$ due to Theorem 1. Since $\text{vec}(\mathbf{R}_{\mathbb{S}} - p_n \mathbf{I}) \in \mathcal{CS}$, as in (6), and $p_n \text{vec}(\mathbf{I}) \in \mathcal{CS}$, we have $\text{vec}(\mathbf{R}_{\mathbb{S}}) \in \mathcal{CS}$. Hence, in the finite snapshot scenario, we can find the vector $\mathbf{p}^\star$ in $\mathcal{CS}$ that minimizes the Euclidean distance to $\text{vec}(\widetilde{\mathbf{R}}_{\mathbb{S}})$. This idea can be formally expressed as the following covex program

$$\text{(P2): } \mathbf{p}^\star \triangleq \arg\min_{\mathbf{p}} \ \|\text{vec}(\widetilde{\mathbf{R}}_{\mathbb{S}}) - \mathbf{p}\|_2^2 \tag{21}$$

$$\text{subject to } (\mathbf{I} - \mathbf{J}\mathbf{J}^\dagger)\mathbf{p} = \mathbf{0}. \tag{22}$$

Here we Note that (P2) is close to, but different from (P1) in several ways. First, the cost function in (P1) is *the spectral norm of a matrix* while that in (P2) is the Euclidean norm of the vector $\text{vec}(\widetilde{\mathbf{R}}_{\mathbb{S}}) - \mathbf{p}$, which is equivalent to *the Frobenius norm of the matrix* $\widetilde{\mathbf{R}}_{\mathbb{S}} - \mathbf{P}$ such that $\mathbf{p} = \text{vec}(\mathbf{P})$. Second, in (P1), the signal term and the noise term are handled separately while in (P2), the vector $\mathbf{p}^\star$ contains the information of sources and noise. Finally, the positive semidefinite constraint (13) is dropped in (P2).

The solution to (P2) is given by

$$\mathbf{p}^\star = \mathbf{J}\mathbf{J}^\dagger \text{vec}(\widetilde{\mathbf{R}}_{\mathbb{S}}). \tag{23}$$

Note that (23) can be evaluated directly, given the sample covariance matrix $\widetilde{\mathbf{R}}_{\mathbb{S}}$ and the array configuration. The computational complexity of (23) is much less than solving (P2) numerically.

Alternatively, Eq. (23) can be written as

$$\mathbf{p}^\star = \mathbf{J}\widetilde{\mathbf{x}}_{\mathbb{D}} \quad \in \mathbb{C}^{|\mathbb{S}|^2}, \quad \widetilde{\mathbf{x}}_{\mathbb{D}} \triangleq \mathbf{J}^\dagger \text{vec}(\widetilde{\mathbf{R}}_{\mathbb{S}}) \quad \in \mathbb{C}^{|\mathbb{D}|}. \tag{24}$$

Due to (24) and Lemma 2, the sample value of $\widetilde{\mathbf{x}}_{\mathbb{D}}$ at the coarray location $m \in \mathbb{D}$ is given by

$$\langle \widetilde{\mathbf{x}}_{\mathbb{D}} \rangle_m = \frac{1}{w(m)} \sum_{n_1 - n_2 = m} \langle \widetilde{\mathbf{R}}_{\mathbb{S}} \rangle_{n_1, n_2}, \tag{25}$$

where $n_1, n_2 \in \mathbb{S}$. Eq. (25) was previously known as ***redundancy averaging*** [7] and [15, Definition 3]. The vector $\widetilde{\mathbf{x}}_{\mathbb{D}}$ is known to be the sample autocorrelation vector on the difference coarray, which was used extensively in coarray-based DOA estimators [7], [8], [15] These arguments show that, *redundancy averaging is closely related to (P2), which uses the concept of the correlation subspace*.

Another related technique is ***rectification***, described in [14] and the references therein. Rectification is composed of two steps. First, a subspace $\mathcal{S}$ is determined so that 1) $\text{vec}(\mathbf{I}) \in \mathcal{S}$ and 2) $\mathcal{S}$ has the minimum distance to the set $\{\mathbf{c}(\bar{\theta})/|\mathbb{S}| : -1/2 \le \bar{\theta} \le 1/2\}$, where the correlation vector $\mathbf{c}(\cdot)$ is defined in (4). Second, the sample covariance matrix $\widetilde{\mathbf{R}}_{\mathbb{S}}$ is rectified by computing the orthogonal projection of $\widetilde{\mathbf{R}}_{\mathbb{S}}$ onto $\mathcal{S}$. This step helps to improve the estimation performance [14]. However, the connection between $\mathcal{S}$ and the correlation subspace $\mathcal{CS}$, in which $\text{vec}(\mathbf{R}_{\mathbb{S}} - p_n \mathbf{I})$ resides, remains unclear in the literature. Under the assumptions in this paper, we can show that $\mathcal{S} = \mathcal{CS}$, implying that *rectification is equivalent to redundancy averaging*. This unifies the theory of the correlation subspace [12], redundancy averaging [7], and rectification [14].

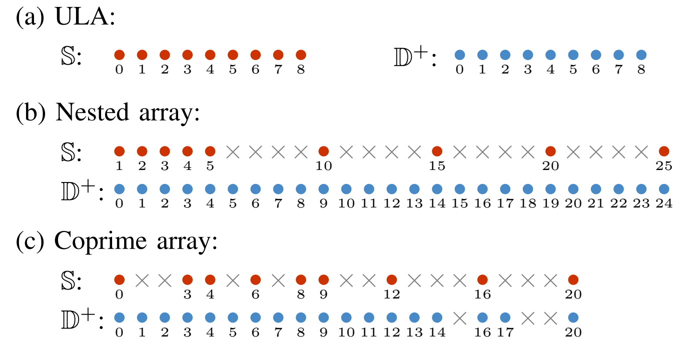

Fig. 1. The sensor locations $\mathbb{S}$ and the nonnegative part of the difference coarrays $\mathbb{D}^+$ for (a) ULA with 9 sensors, (b) the nested array with $N_1 = 4, N_2 = 5$, and (c) the coprime array with $M = 3, N = 4$.

V. Numerical Examples

In this section, we will demonstrate the eigenvalues of $\mathbf{S}(\rho_2)$, which were theoretically studied in (18). Here two methods of computing the eigenvalues of $\mathbf{S}(\rho_2)$ are used. Method 1 is based on the numerical integration and the numerical eigen-decomposition, as used in [12]. Due to (18), Method 2 computes the weight functions using Definition 5. It will be shown that Method 2 gives the same result as Method 1. However, Method 2 is faster and more insightful, as we discussed in Section III.

Consider the following three 1D array configurations: the ULA with 9 sensors, the nested array with $N_1 = 4, N_2 = 5$, and the coprime array with $M = 3, N = 4$, where the notations are in accordance with [8], [9]. The number of sensors is 9 for each array. The sensor locations and the nonnegative part of the difference coarrays for these arrays are depicted in Fig. 1. Since the difference coarray is symmetric, the size of the difference coarray is 17 for ULA, 49 for the nested array, and 35 for the coprime array.

For a fixed array configuration, the details of Method 1 are given as follows. We first compute the numerical approximation of $\mathbf{S}(\rho_2)$, denoted by $\widetilde{\mathbf{S}}(\rho_2)$, as follows:

$$\widetilde{\mathbf{S}}(\rho_2) = \sum_{\ell = -(N_{\text{pt}}-1)/2}^{(N_{\text{pt}}-1)/2} \mathbf{c}(\ell\Delta)\mathbf{c}^H(\ell\Delta)\rho_2(\ell\Delta) \times \Delta, \tag{26}$$

where the number of discrete samples is $N_{\text{pt}} = 2^{14} + 1$ and the step size is $\Delta = 1/N_{\text{pt}}$. Then the eigenvalues of $\widetilde{\mathbf{S}}(\rho_2)$ are computed numerically [12]. These results are plotted in Fig. 2(a), (c), and (e). Method 2 calculates the weight functions based on Definition 5, as shown in Fig. 2(b), (d), and (f).

It can be observed from Fig. 2(a), (c), and (e) that the number of positive eigenvalues that are away from zero, is 17

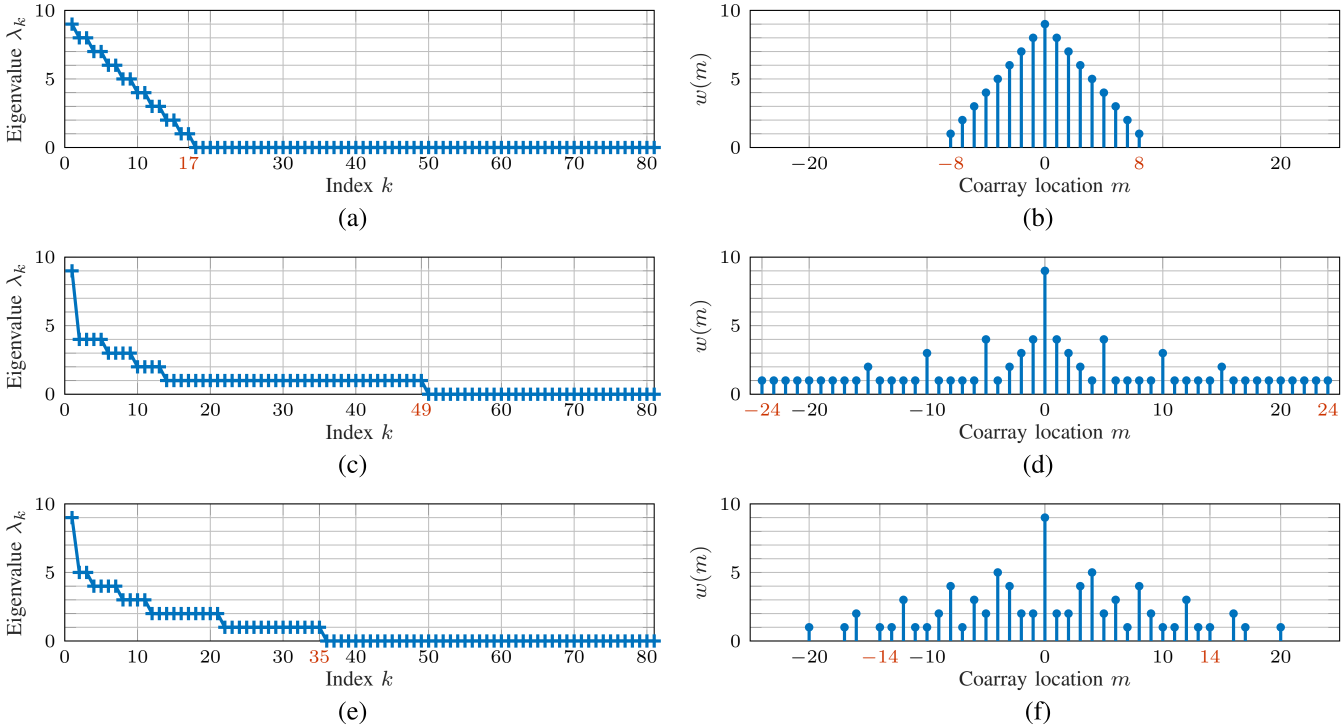

Fig. 2. The eigenvalues of the matrix $\widetilde{\mathbf{S}}(\rho_2)$ (Method 1, left) and the weight functions (Method 2, right) for (a), (b) the ULA with 9 sensors ($|\mathbb{D}| = 17$), (c), (d) the nested array with $N_1 = 4, N_2 = 5$ (9 sensors, $|\mathbb{D}| = 49$), and (e), (f) the coprime array with $M = 3, N = 4$ (9 sensors, $|\mathbb{D}| = 35$). Here the matrices $\widetilde{\mathbf{S}}(\rho_2)$ are given by (26) and the eigenvalues of $\widetilde{\mathbf{S}}(\rho_2)$ are obtained numerically.

for ULA, 49 for the nested array, and 35 for the coprime array. These results are consistent with the size of the difference coarray. Furthermore, the eigenvalues for $\widetilde{\mathbf{S}}(\rho_2)$ (Method 1) coincide with the weight functions (Method 2). For instance, in Fig. 2(e), the eigenvalues $\lambda_2 = \lambda_3 = 5$ while, in Fig. 2(f), the weight functions satisfy $w(4) = w(-4) = 5$.

VI. Concluding Remarks

In this paper, we proposed closed-form expressions for the correlation subspace, which reveal a fundamental connection between the correlation subspace and the difference coarray. Our results not only simplify the computation of the correlation subspace greatly, but also show an interesting logical connection between the correlation subspace, redundancy averaging, and rectification. In the future, it would be of considerable interest to exploit the (generalized) correlation subspace to the case of prior knowledge about sources [18], multiple dimensions [18], and even state-of-the-art DOA estimators like SPICE [6] and atomic norm based methods [19].

References

[1] R. N. Bracewell, *Radio Astronomy Techniques*. Berlin: Springer, 1962, vol. 54.
[2] M. I. Skolnik, *Introduction to Radar Systems*, 3rd ed. McGraw Hill, 2001.
[3] H. L. Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*. Hoboken, NJ, USA: Wiley, 2002.
[4] R. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, no. 3, pp. 276–280, March 1986.
[5] R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 37, no. 7, pp. 984–995, July 1989.
[6] P. Stoica, P. Babu, and J. Li, "SPICE: A sparse covariance-based estimation method for array processing," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 629–638, Feb. 2011.
[7] S. Pillai, Y. Bar-Ness, and F. Haber, "A new approach to array geometry for improved spatial spectrum estimation," *Proc. IEEE*, vol. 73, no. 10, pp. 1522–1524, Oct. 1985.
[8] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.
[9] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.
[10] Y. D. Zhang, M. G. Amin, and B. Himed, "Sparsity-based DOA estimation using co-prime arrays," in *Proc. IEEE Int. Conf. Acoust., Speech, and Sig. Proc.*, May 2013, pp. 3967–3971.
[11] Z. Tan, Y. Eldar, and A. Nehorai, "Direction of arrival estimation using co-prime arrays: A super resolution viewpoint," *IEEE Trans. Signal Process.*, vol. 62, no. 21, pp. 5565–5576, Nov. 2014.
[12] M. Rahmani and G. K. Atia, "A subspace method for array covariance matrix estimation," in *Proc. IEEE Sensor Array and Multichannel Signal Process. Workshop*, Rio de Janeiro, Brazil, July 2016, pp. 1–5.
[13] R. T. Hoctor and S. A. Kassam, "The unifying role of the coarray in aperture synthesis for coherent and incoherent imaging," *Proc. IEEE*, vol. 78, no. 4, pp. 735–752, April 1990.
[14] P. Forster, "Generalized rectification of cross spectral matrices for arrays of arbitrary geometry," *IEEE Trans. Signal Process.*, vol. 49, no. 5, pp. 972–978, May 2001.
[15] C.-L. Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," *IEEE Signal Process. Lett.*, vol. 22, no. 9, pp. 1438–1442, Sept. 2015.
[16] ——, "Cramér-Rao bounds for coprime and other sparse arrays, which find more sources than sensors," *Digit. Signal Process.*, vol. 61, pp. 43–61, Feb. 2017, special Issue on Coprime Sampling and Arrays.
[17] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, 1985.
[18] C.-L. Liu and P. P. Vaidyanathan, "Correlation subspaces: Generalizations and connection to difference coarrays," *IEEE Trans. Signal Process.*, vol. 65, no. 19, pp. 5006–5020, Oct. 2017.
[19] B. N. Bhaskar, G. Tang, and B. Recht, "Atomic norm denoising with applications to line spectral estimation," *IEEE Trans. Signal Process.*, vol. 61, no. 23, pp. 5987–5999, Dec. 2013.