3D Rotation Estimation Using Discrete Spherical Harmonic Oscillator Transforms

Soo-Chang Pei
pei@cc.ee.ntu.edu.tw

National Taiwan University
Department of Electrical Engineering
No. 1, Sec. 4 Roosevolt Rd.
Taipei, Taiwan.

Chun-Lin Liu
cl.liu@caltech.edu

California Institute of Technology
Department of Electrical Engineering
1200 E. California Blvd.,
Pasadena, CA 91125, USA.

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Outline

1. Introduction: 3D Rotation Angle Estimation Problem
2. Rotation Angle estimation Using Discrete Spherical Harmonic Oscillator Transforms
    - Discrete Spherical Harmonic Oscillator Transforms
    - Object Rotation
3. Experimental Results
4. Conclusion

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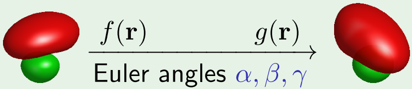
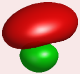
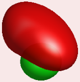
3D Rotation Angle Estimation

Underlying Model



$$f(\mathbf{r}) \xrightarrow[\text{Euler angles } \alpha, \beta, \gamma]{} g(\mathbf{r})$$

Problem Statement

Given $f(\mathbf{r})$  and $g(\mathbf{r})$ , estimate Euler angles $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$.

- Applications in
  - pattern recognition,[1]
  - computer vision,[2]
  - robotics,[3] and
  - computerized tomography imaging.[4]

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[1] M. Kazhdan. “An Approximate and Efficient Method for Optimal Rotation Alignment of 3D Models”. In: *IEEE Trans. Pattern Anal. Mach. Intell.* 29.7 (2007), pp. 1221–1229.

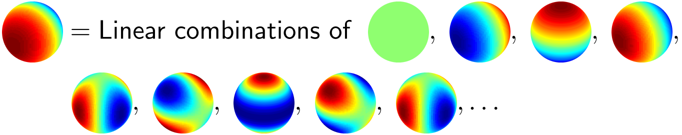
[2] Dietmar Saupe and DejanV. Vranić. “3D Model Retrieval with Spherical Harmonics and Moments”. In: *Pattern Recogn.* Vol. 2191. 2001, pp. 392–397.

[3] A. Makadia, L. Sorgi, and K. Daniilidis. “Rotation Estimation from Spherical Images”. In: *Proc. of 2004 ICPR*. vol. 3. 2004, 590–593 Vol.3.

[4] E.J. Garboczi. “Three-dimensional mathematical analysis of particle shape using X-ray tomography and spherical harmonics: Application to aggregates used in concrete ”. In: *Cement Concrete Res.* 32.10 (2002), pp. 1621–1638.

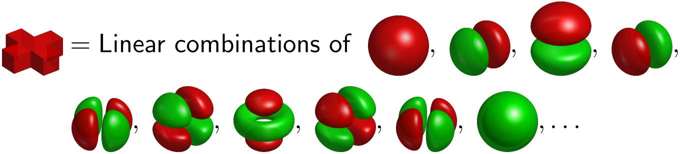
Transform-Domain Methods

- Spherical Harmonic Transforms (SHTs)[5] : Expand 2D surface signal $f(\theta, \varphi)$ in terms of spherical harmonics, $Y_{\ell,m}(\theta, \varphi)$, with coefficients $F_{\ell,m}$.



$=$ Linear combinations of , , , , , , , , , …

- Spherical Fourier Transforms (SFTs)[6] : Express 3D signal $f(\mathbf{r})$ in terms of some basis functions, yielding the coefficients $S_{n\ell m}$.



$=$ Linear combinations of , , , , , , , , , , …

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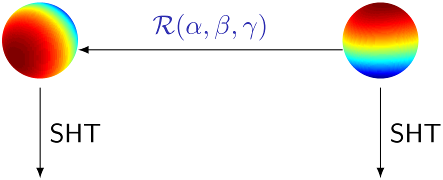
[5] Eugene P. Wigner. *Group theory and its application to quantum mechanics of atom spectra*. Academic Press, 1959.

[6] Qing Wang, O. Ronneberger, and H. Burkhardt. "Rotational Invariance Based on Fourier Analysis in Polar and Spherical Coordinates". In: *IEEE Trans. Pattern Anal. Mach. Intell.* 31.9 (2009), pp. 1715–1722.

Rotation and Wigner D-matrices

Spatial Domain:



SHT Domain:

$$\begin{bmatrix} F_{1,1}(\alpha, \beta, \gamma) \\ F_{1,0}(\alpha, \beta, \gamma) \\ F_{1,-1}(\alpha, \beta, \gamma) \end{bmatrix} = \mathbf{D}^{\ell=1}(\alpha, \beta, \gamma) \begin{bmatrix} F_{1,1} \\ F_{1,0} \\ F_{1,-1} \end{bmatrix}$$

- Convert spatial rotations to matrix multiplications.
- Estimate α, β, γ from SHTs.

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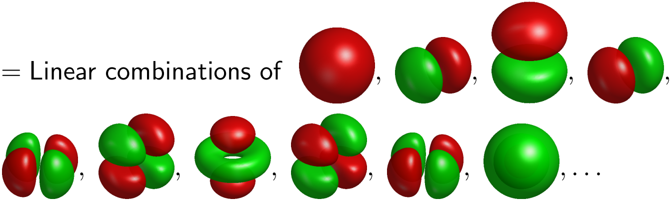
Spherical Harmonics Oscillator Transforms (SHOTs)[7,8]

- Expand 3D signal $f(\mathbf{r})$ onto Spherical Harmonic Oscillator Wavefunctions (SHOWs), $\langle \mathbf{r} | n\ell m) = N_{n\ell} r^{\ell} L_n^{\ell+1/2}\left(r^2\right) e^{-r^2/2} Y_{\ell m}(\theta, \varphi)$,

$$(n\ell m | f\rangle = \int f(\mathbf{r}) \langle \mathbf{r} | n\ell m)^* \, \mathrm{d}^3\mathbf{r},$$

$$f(\mathbf{r}) = \sum_{n,\ell \in \mathbb{Z}_+} \sum_{m=-\ell}^{\ell} \langle \mathbf{r} | n\ell m) (n\ell m | f\rangle,$$

$=$ Linear combinations of



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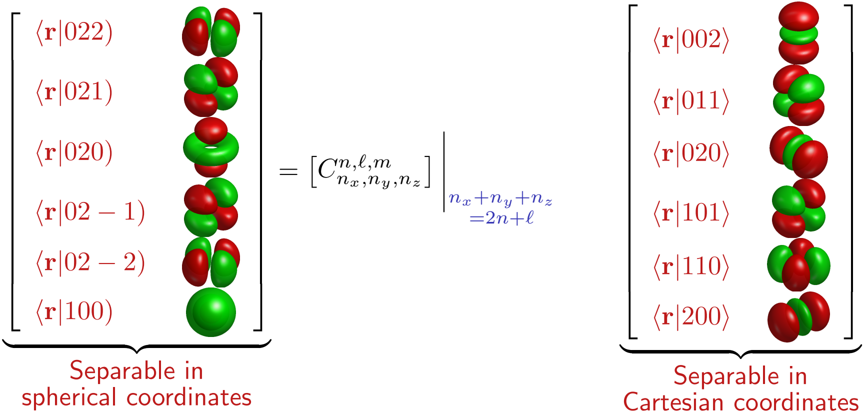
[7] David J. Griffiths. *Introduction to Quantum Mechanics*. 2nd ed. Pearson Prentice Hall, 2004.

[8] Soo-Chang Pei and Chun-Lin Liu. "Discrete spherical harmonic oscillator transforms on the Cartesian grids using transformation coefficients". In: *IEEE Trans. Signal Process.* 61.5 (2013), pp. 1149 –1164.

Transformation Coefficients

- SHOWs are linear combinations of 3D separable Hermite Gaussian functions.

$$\langle \mathbf{r}|n\ell m) = \sum_{n_x+n_y+n_z=2n+\ell} C^{n,\ell,m}_{n_x,n_y,n_z} \quad \langle \mathbf{r}|n_x n_y n_z\rangle$$



$$\underbrace{\begin{bmatrix} \langle \mathbf{r}|022) \\ \langle \mathbf{r}|021) \\ \langle \mathbf{r}|020) \\ \langle \mathbf{r}|02-1) \\ \langle \mathbf{r}|02-2) \\ \langle \mathbf{r}|100) \end{bmatrix}}_{\text{Separable in spherical coordinates}} = \left[C^{n,\ell,m}_{n_x,n_y,n_z}\right]\Bigg|_{\substack{n_x+n_y+n_z\\=2n+\ell}} \underbrace{\begin{bmatrix} \langle \mathbf{r}|002\rangle \\ \langle \mathbf{r}|011\rangle \\ \langle \mathbf{r}|020\rangle \\ \langle \mathbf{r}|101\rangle \\ \langle \mathbf{r}|110\rangle \\ \langle \mathbf{r}|200\rangle \end{bmatrix}}_{\text{Separable in Cartesian coordinates}}$$

Advantages of discrete SHOTs

Discrete SHOWs = 3D separable Discrete Hermite Gaussian functions
and Transformation Coefficients,

Discrete SHOTs = 3D Hermite transforms
and Transformation Coefficients.

- Discrete SHOWs are defined on Cartesian grids.
- Spherical components can be analyzed directly from volume data without interpolation.
- In Cartesian coordinates, decompose nonseparable transforms (SHOTs) into separable transforms (3D Hermite transforms).
- Fast Hermite transforms.[9]
- Fast transformation coefficient evaluation.[10]

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[9] Gregory Leibona et al. "A fast Hermite transform". In: *Theor. Comput. Sci.* 409.2 (2008), 211228.

[10] Soo-Chang Pei and Chun-Lin Liu. "Discrete spherical harmonic oscillator transforms on the Cartesian grids using transformation coefficients". In: *IEEE Trans. Signal Process.* 61.5 (2013), pp. 1149 –1164.

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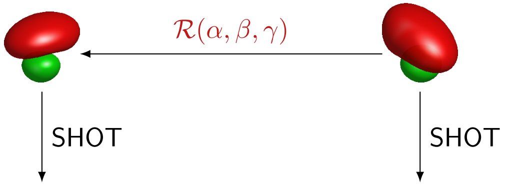
Contribution: Object Rotation on Discrete SHOTs

Spatial Domain:

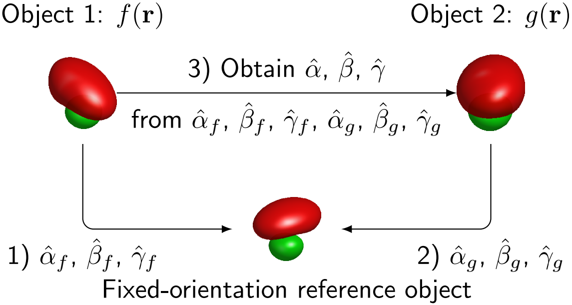


SHT Domain:

$$\begin{bmatrix} (n11|f\rangle\,(\alpha,\beta,\gamma) \\ (n10|f\rangle\,(\alpha,\beta,\gamma) \\ (n1-1|f\rangle\,(\alpha,\beta,\gamma) \end{bmatrix} = \mathbf{D}^{\ell=1}(\alpha,\beta,\gamma) \begin{bmatrix} (n11|f\rangle \\ (n10|f\rangle \\ (n1-1|f\rangle \end{bmatrix}$$

- Exactly the same as that in the SHT case.
- Any angle estimation algorithm based on Wigner-D matrices is compatible with SHOTs.

Angle Estimation Algorithms from Wigner-D matrices[11]



Object 1: $f(\mathbf{r})$

Object 2: $g(\mathbf{r})$

3) Obtain $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$

from $\hat{\alpha}_f$, $\hat{\beta}_f$, $\hat{\gamma}_f$, $\hat{\alpha}_g$, $\hat{\beta}_g$, $\hat{\gamma}_g$

1) $\hat{\alpha}_f$, $\hat{\beta}_f$, $\hat{\gamma}_f$

2) $\hat{\alpha}_g$, $\hat{\beta}_g$, $\hat{\gamma}_g$

Fixed-orientation reference object

$$\begin{aligned}
\hat{\alpha}_f &= \sphericalangle \, (n_0 11 | f\rangle \, , \\
\hat{\beta}_f &= \arctan \left(- \frac{\sqrt{2} \, (n_0 11 | f\rangle \, (\hat{\alpha}_f, 0, 0)}{(n_0 10 | f\rangle \, (\hat{\alpha}_f, 0, 0)} \right) , \\
\hat{\gamma}_f &= \sphericalangle \, (n_0 21 | f\rangle \left(\hat{\alpha}_f, \hat{\beta}_f, 0 \right) ,
\end{aligned}$$

$$\mathbf{R}_{\hat{\alpha},\hat{\beta},\hat{\gamma}} = \mathbf{R}^{-1}_{\hat{\alpha}_g,\hat{\beta}_g,\hat{\gamma}_g} \mathbf{R}_{\hat{\alpha}_f,\hat{\beta}_f,\hat{\gamma}_f} .$$

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[11] Gilles Burel and Hugues Hénocq. "Determination of the orientation of 3D objects using spherical harmonics". In: *Graph. Models Image Process.* 57.5 (1995), pp. 400–408.

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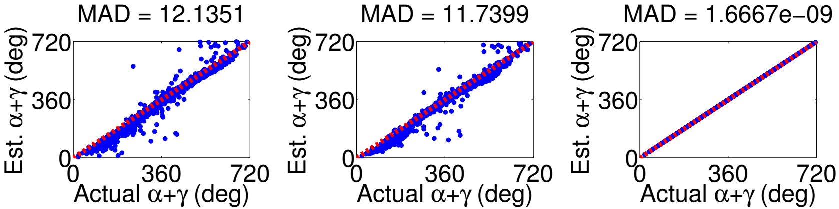
Parameter Settings

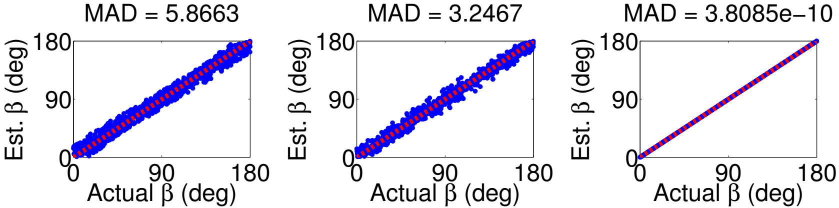
- The continuous signal: three Gaussian mixures with different gains, centers, and variances.
- Take 31 samples in each dimension from the continuous signal and its rotated version.
- The test signal can be considered to be nearly bandlimited signals.
- Running time in seconds:

	SHT	SFT	SHOT
Test 1	1.566143	3.725554	3.669643
Test 2	1.281753	3.370138	3.559185
Test 3	1.276130	3.361931	3.532833

Simulation 1: Estimation Bias and Variance

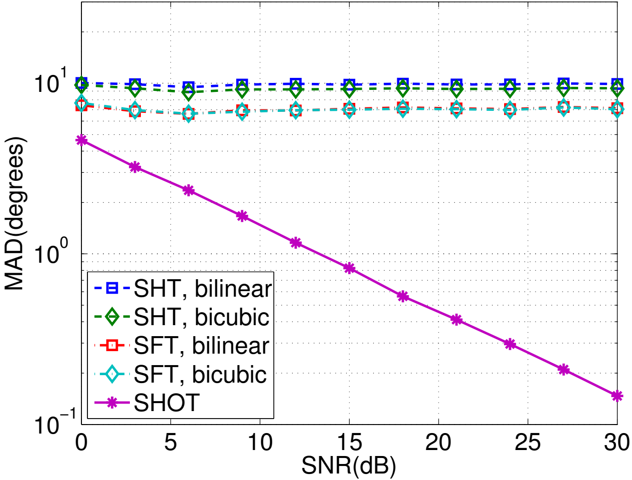


(a) SHT (b) SFT (c) SHOT



(d) SHT (e) SFT (f) SHOT

Simulation 2: Noise Tolerance



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Discussion and Conclusion

- We presented a 3D rotation estimation algorithm based on discrete SHOTs.
- Advantages of discrete SHOTs:
  1. Fast computation algorithms.
  2. Compatible with spherical-harmonic-based angle estimation algorithms.
- Given volume data, bandlimited inputs, our method outperforms spherical harmonic transforms[12] and spherical Fourier transforms[13] in terms of estimation variance and noise tolerance.
- Applications in
  - 3D object alignment,
  - 3D object registration,
  - 3D object retrieval.

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[12] Gilles Burel and Hugues Hénocq. "Determination of the orientation of 3D objects using spherical harmonics". In: *Graph. Models Image Process.* 57.5 (1995), pp. 400–408.

[13] Qing Wang, O. Ronneberger, and H. Burkhardt. "Rotational Invariance Based on Fourier Analysis in Polar and Spherical Coordinates". In: *IEEE Trans. Pattern Anal. Mach. Intell.* 31.9 (2009), pp. 1715–1722.