New Cramér-Rao Bound Expressions for Coprime and Other Sparse Arrays

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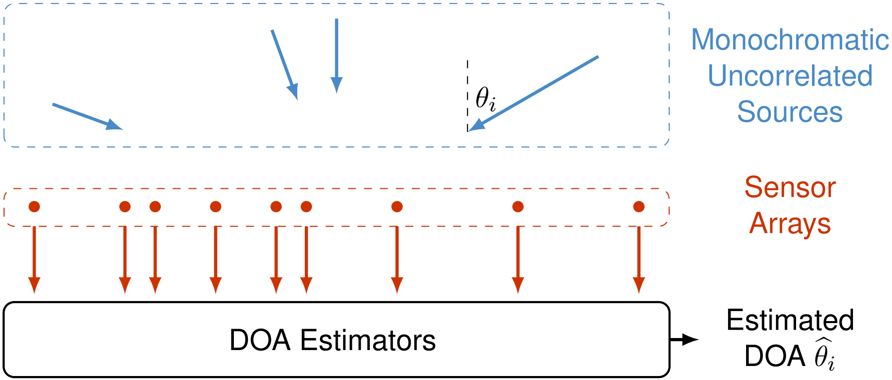
Outline

1. DOA Estimation using Sparse Arrays
2. Cramér-Rao Bounds for Sparse Arrays
3. Numerical Examples
4. Concluding Remarks

Outline

Direction-of-arrival (DOA) estimation[1]



[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

ULA and sparse arrays

ULA (not sparse)

- Identify at most $N-1$ uncorrelated sources, given N sensors.[1]
- Can only find fewer sources than sensors.

Sparse arrays

1. Minimum redundancy arrays[2]
2. Nested arrays[3]
3. Coprime arrays[4]
4. Super nested arrays[5]

- Identify $O(N^2)$ uncorrelated sources with $O(N)$ physical sensors.
- More sources than sensors!

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968.

[3] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010.

[4] Vaidyanathan and Pal, *IEEE Trans. Signal Proc.*, 2011.

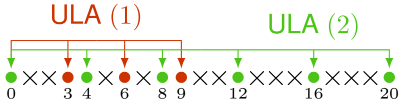
[5] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Coprime arrays[1]

The coprime array with $(M, N) = 1$ is the union of

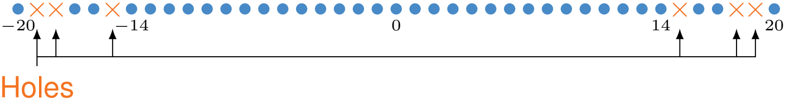
1. an N-element ULA with spacing $M\lambda/2$ and
2. a $2M$-element ULA with spacing $N\lambda/2$.

Physical array $\mathbb{S}$ ($M = 3, N = 4$):



Difference coarray

$$\mathbb{D} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{S}\}$$



[1] Vaidyanathan and Pal, *IEEE Trans. Signal Proc.*, 2011.

DOA estimation using sparse arrays

Multiple data vectors in the physical array $\mathbb{S}$
$|\mathbb{S}| = O(N)$

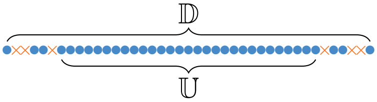
Step 1 →

A single correlation vector in the difference array $\mathbb{D}$
$|\mathbb{D}| = O(N^2)$

Step 2: DOA estimation using this single correlation vector.

- Spatial smoothing MUSIC (SS MUSIC),[1] to name a few[2,3].

Empirically, SS MUSIC can identify up to $(|\mathbb{U}| - 1)/2$ uncorrelated sources with sufficient snapshots. **Why?**



[1] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010.
[2] Abramovich, Spencer, and Gorokhov, *IEEE Trans. Signal Proc.*, 1998, 1999.
[3] Zhang, Amin, and Himed, *IEEE ICASSP*, 2013; Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2015;

Outline

Fisher Information Matrices and Cramér-Rao Bounds

- $\mathbf{x}$: random vectors with distribution $p(\mathbf{x}; \boldsymbol{\alpha})$.
- $\boldsymbol{\alpha}$: Real-valued, unknown, and deterministic parameters.

- FIM $\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})$, based on $p(\mathbf{x}; \boldsymbol{\alpha})$ and $\boldsymbol{\alpha}$. (under some regularity conditions)
- If $\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})$ is nonsingular, then $\mathrm{CRB}(\boldsymbol{\alpha}) \triangleq \boldsymbol{\mathcal{I}}^{-1}(\boldsymbol{\alpha})$.

> If $\widehat{\boldsymbol{\alpha}}(\mathbf{x})$ is a unbiased estimator of $\boldsymbol{\alpha}$, then $\mathrm{Cov}(\widehat{\boldsymbol{\alpha}}(\mathbf{x})) \succeq \mathrm{CRB}(\boldsymbol{\alpha})$.

- CRB depend only on $p(\mathbf{x}; \boldsymbol{\alpha})$ and $\boldsymbol{\alpha}$, NOT on the estimators $\widehat{\boldsymbol{\alpha}}(\mathbf{x})$.
- CRB pose fundamental limits to the estimation performance of *all* unbiased estimators.

Related work

1. Deterministic CRB[1] assumes $A_i(k)$ are deterministic.
2. Stochastic CRB[2] becomes invalid for $D > |\mathbb{S}|$ since $\mathbf{V}_{\mathbb{S}}^H \mathbf{V}_{\mathbb{S}}$ is singular.
   $\mathrm{CRB} = \frac{p_n}{2K} \mathrm{Re} \left[\mathbf{H} \odot \mathbf{Q}^T \right]^{-1}$
   $\mathbf{H} = \mathbf{U}_{\mathbb{S}}^H [\mathbf{I} - \mathbf{V}_{\mathbb{S}} (\mathbf{V}_{\mathbb{S}}^H \mathbf{V}_{\mathbb{S}})^{-1} \mathbf{V}_{\mathbb{S}}^H] \mathbf{U}_{\mathbb{S}}$.
3. Abramovich *et al.*[3] plotted the CRB curves numerically.
4. Jansson *et al.*'s CRB expressions[4] are undefined if $D = |\mathbb{S}|$.

$$\mathbf{x}_{\mathbb{S}}(k) = \sum_{i=1}^{D} A_i(k) \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathbb{S}}(k)$$

$$\mathbf{V}_{\mathbb{S}} = [\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)] \in \mathbb{C}^{|\mathbb{S}| \times D}$$

SS MUSIC

- Sources are stochastic and known to be uncorrelated.
  $(\mathbb{E}[A_i(k_1) A_j^*(k_2)] = p_i \delta_{i,j} \delta_{k_1,k_2})$
- Resolve more sources than sensors ($D \geq |\mathbb{S}|$).

[1] Stoca and Nehorai, *IEEE Trans. Acoustics, Speech and Signal Proc.*, 1989.
[2] Stoca and Nehorai, *IEEE Trans. Acoustics, Speech and Signal Proc.*, 1990.
[3] Abramovich, Gray, Gorokhov, and Spencer, *IEEE Trans. Signal Proc.*, 1998.
[4] Jansson, Göransson, and Ottersten, *IEEE Trans. Signal Proc.*, 1999.

Augmented Coarray Manifold matrices (*Proposed*)

$$\mathbf{A}_c = \begin{bmatrix} \operatorname{diag}(\mathbb{D})\mathbf{V}_{\mathbb{D}} & \mathbf{V}_{\mathbb{D}} & \mathbf{e}_0 \end{bmatrix}$$

$\mathbb{D} = \{-4, -3, -1, 0, 1, 3, 4\}$. $D = 2$ sources $\bar{\theta}_1$ and $\bar{\theta}_2$.

$$\underbrace{\overbrace{\begin{bmatrix} (-4)e^{j2\pi\bar{\theta}_1(-4)} & (-4)e^{j2\pi\bar{\theta}_2(-4)} \\ (-3)e^{j2\pi\bar{\theta}_1(-3)} & (-3)e^{j2\pi\bar{\theta}_2(-3)} \\ (-1)e^{j2\pi\bar{\theta}_1(-1)} & (-1)e^{j2\pi\bar{\theta}_2(-1)} \\ (0)e^{j2\pi\bar{\theta}_1(0)} & (0)e^{j2\pi\bar{\theta}_2(0)} \\ (1)e^{j2\pi\bar{\theta}_1(1)} & (1)e^{j2\pi\bar{\theta}_2(1)} \\ (3)e^{j2\pi\bar{\theta}_1(3)} & (3)e^{j2\pi\bar{\theta}_2(3)} \\ (4)e^{j2\pi\bar{\theta}_1(4)} & (4)e^{j2\pi\bar{\theta}_2(4)} \end{bmatrix}}^{D}}_{\operatorname{diag}(\mathbb{D})\mathbf{V}_{\mathbb{D}}} \quad \underbrace{\overbrace{\begin{bmatrix} e^{j2\pi\bar{\theta}_1(-4)} & e^{j2\pi\bar{\theta}_2(-4)} \\ e^{j2\pi\bar{\theta}_1(-3)} & e^{j2\pi\bar{\theta}_2(-3)} \\ e^{j2\pi\bar{\theta}_1(-1)} & e^{j2\pi\bar{\theta}_2(-1)} \\ e^{j2\pi\bar{\theta}_1(0)} & e^{j2\pi\bar{\theta}_2(0)} \\ e^{j2\pi\bar{\theta}_1(1)} & e^{j2\pi\bar{\theta}_2(1)} \\ e^{j2\pi\bar{\theta}_1(3)} & e^{j2\pi\bar{\theta}_2(3)} \\ e^{j2\pi\bar{\theta}_1(4)} & e^{j2\pi\bar{\theta}_2(4)} \end{bmatrix}}^{D}}_{\mathbf{V}_{\mathbb{D}}} \quad \underbrace{\overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{1}}_{\mathbf{e}_0} \quad |\mathbb{D}|$$

The *proposed* CRB expression for sparse arrays[1]

$\mathbf{A}_c$ has full column rank. $(\mathrm{rank}(\mathbf{A}_c) = 2D+1)$

if and only if

FIM is nonsingular. (CRB exists)

$$\mathrm{CRB}(\bar{\boldsymbol{\theta}}) = \mathrm{CRB}(\bar{\theta}_1, \ldots, \bar{\theta}_D) = \frac{1}{4\pi^2 K}\left(\mathbf{G}_0^H \mathbf{\Pi}^{\perp}_{\mathbf{M}\mathbf{W}_{\mathbb{D}}} \mathbf{G}_0\right)^{-1}.$$

- $\{\bar{\theta}_i\}_{i=1}^{D}$: normalized DOAs. $\bar{\theta}_i = 0.5 \sin\theta_i \in [-0.5, 0.5]$.
- $\mathbb{S}$: The physical array.
- $\mathbb{D}$: The difference coarray.
- p_i: The ith source power.
- p_n: The noise power.
- K: Number of snapshots.

- $\mathbf{V}_{\mathbb{D}} = [\mathbf{v}_{\mathbb{D}}(\bar{\theta}_1), \ldots, \mathbf{v}_{\mathbb{D}}(\bar{\theta}_D)]$.
- $\mathbf{W}_{\mathbb{D}} = [\mathbf{V}_{\mathbb{D}}, \mathbf{e}_0]$.
- $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_D)$.
- $\mathbf{G}_0 = \mathbf{M}\mathrm{diag}(\mathbb{D})\mathbf{V}_{\mathbb{D}}\mathbf{P}$.
- $\mathbf{M} = (\mathbf{J}^H(\mathbf{R}_{\mathbb{S}}^T \otimes \mathbf{R}_{\mathbb{S}})^{-1}\mathbf{J})^{1/2} \succ \mathbf{0}$.
- $\mathbf{R}_{\mathbb{S}} = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + p_n \mathbf{I}$.

- $\mathbf{J} \in \{0,1\}^{|\mathbb{S}|^2 \times |\mathbb{D}|}$, $\langle \mathbf{J} \rangle_{:,m} = \mathrm{vec}(\mathbf{I}(m))$, $m \in \mathbb{D}$.
- $\langle \mathbf{I}(m) \rangle_{n_1,n_2} = \begin{cases} 1, & \text{if } n_1 - n_2 = m, \\ 0, & \text{otherwise}, \end{cases}$ $n_1, n_2 \in \mathbb{S}$.

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[1] Liu and Vaidyanathan, *Digital Signal Proc.,Elsevier*, 2016; `http://systems.caltech.edu/dsp/students/clliu/CRB.html`

Implications drawn from the CRB expressions

ACM matrices depend on

- The difference coarray $\mathbb{D}$,
- the normalized DOA $\bar{\boldsymbol{\theta}} = [\bar{\theta}_1, \ldots, \bar{\theta}_D]^T$, and
- the number of sources D.

These factors influence the existence of CRB.

$$\mathbf{A}_c = \begin{bmatrix} \mathrm{diag}(\mathbb{D})\mathbf{V}_{\mathbb{D}} & \mathbf{V}_{\mathbb{D}} & \mathbf{e}_0 \end{bmatrix}$$

$$\mathrm{CRB}(\bar{\boldsymbol{\theta}}) = \frac{1}{4\pi^2 K} \left(\mathbf{G}_0^H \boldsymbol{\Pi}^{\perp}_{\mathbf{M}\mathbf{W}_{\mathbb{D}}} \mathbf{G}_0 \right)^{-1}$$

CRB depend on

- The physical array $\mathbb{S}$,
- the normalized DOA $\bar{\boldsymbol{\theta}} = [\bar{\theta}_1, \ldots, \bar{\theta}_D]^T$,
- the number of sources D,
- the number of snapshots K, and
- the SNR $p_1/p_n, \ldots, p_D/p_n$.

These parameters control the values of CRB.

If $\mathrm{rank}(\mathbf{A}_c) = 2D + 1$, then $\lim_{K \to \infty} \mathrm{CRB}(\bar{\boldsymbol{\theta}}) = \mathbf{0}$.

Asymptotic CRB expressions for large SNR[1]

1. $\mathbf{A}_c$ has full column rank.
2. D uncorrelated sources have equal SNR p/p_n.
3. Large SNR ($p/p_n \to \infty$).

Theorem ($\mathrm{rank}(\mathbf{V}_{\mathbb{S}}) < |\mathbb{S}|$)

$D < |\mathbb{S}|$

$$\mathrm{CRB}(\bar{\boldsymbol{\theta}}) = \frac{1}{4\pi^2 K (p/p_n)} \mathbf{S}_1^{-1},$$

where $\mathbf{S}_1$ is positive definite and independent of SNR.

$$\lim_{p/p_n \to \infty} \mathrm{CRB}(\bar{\boldsymbol{\theta}}) = \mathbf{0}.$$

Theorem ($\mathrm{rank}(\mathbf{V}_{\mathbb{S}}) \geq |\mathbb{S}|$)

Typically $D \geq |\mathbb{S}|$

$$\mathrm{CRB}(\bar{\boldsymbol{\theta}}) = \frac{1}{4\pi^2 K} \mathbf{S}_2^{-1},$$

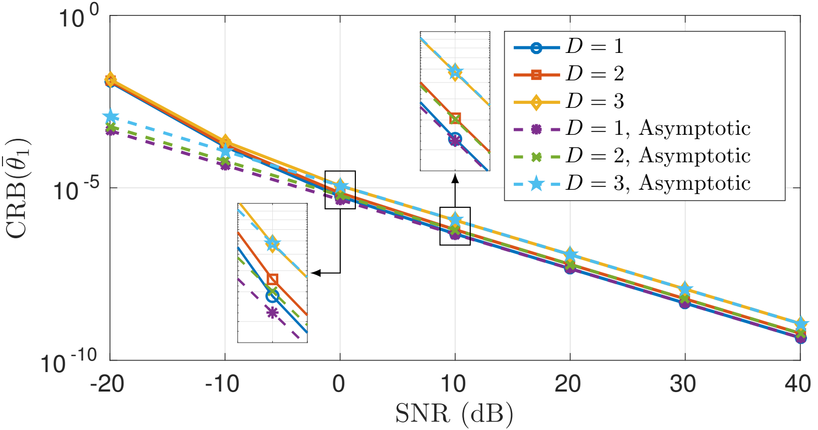
where $\mathbf{S}_2$ is positive definite and independent of SNR.

$$\lim_{p/p_n \to \infty} \mathrm{CRB}(\bar{\boldsymbol{\theta}}) \succ \mathbf{0}.$$

[1] These phenomena were observed by Abramovich et al. in 1998, which can be proved using our CRB expressions.

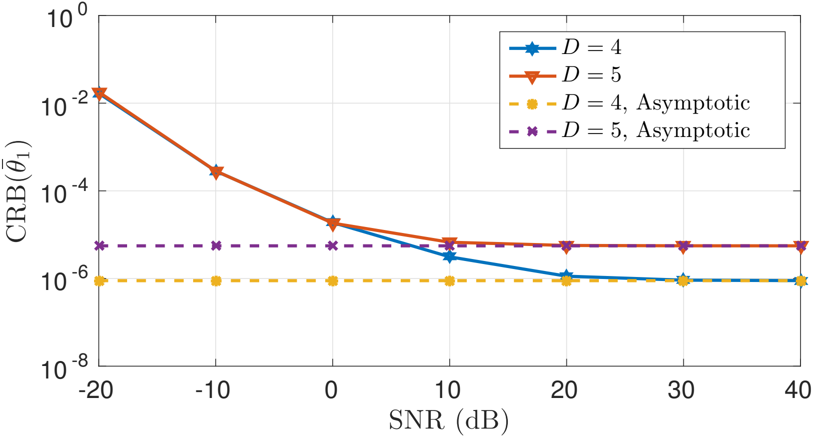
Outline

CRB vs SNR, fewer sources than sensors $(D < |\mathbb{S}| = 4)$



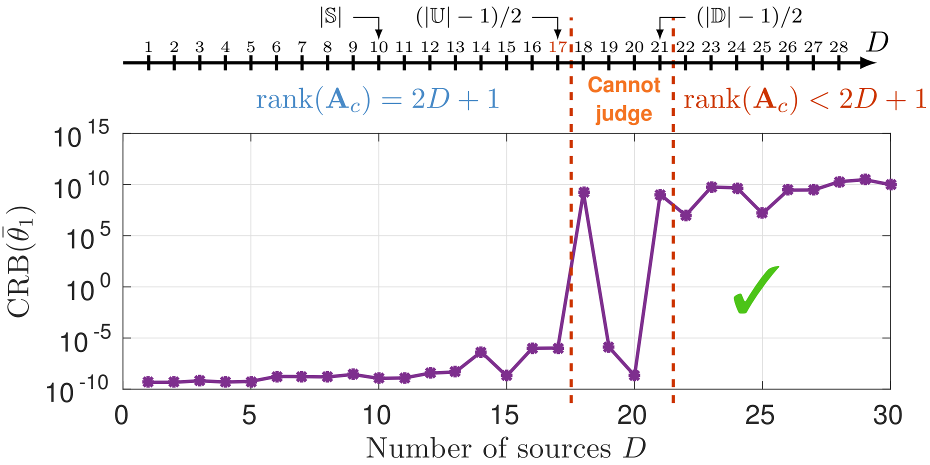
- $\mathbb{S} = \{1, 2, 3, 6\}$.
- $\mathbb{D} = \{-5, \ldots, 5\}$.
- $\bar{\theta}_i = -0.49 + 0.95(i-1)/D$.
- The number of snapshots $K = 200$.

CRB vs SNR, more sources than sensors $(D \geq |\mathbb{S}| = 4)$



- $\mathbb{S} = \{1, 2, 3, 6\}$.
- $\mathbb{D} = \{-5, \ldots, 5\}$.
- $\bar{\theta}_i = -0.49 + 0.95(i-1)/D$.
- The number of snapshots $K = 200$.

CRB vs D, coprime array, 10 sensors[1]



- $M=3, N=5, \mathbb{S}=\{0,3,5,6,9,10,12,15,20,25\}$.
- $\mathbb{D}=\{0,\pm1,\ldots,\pm17,\pm19,\pm20,\pm22,\pm25\}$.
- $\bar{\theta}_i = -0.48 + (i-1)/D$ for $i=1,\ldots,D$. 20dB SNR, 500 snapshots.

[1] Liu and Vaidyanathan, *Digital Signal Proc.,Elsevier*, 2016; `http://systems.caltech.edu/dsp/students/clliu/CRB.html`

Outline

Concluding remarks

- The proposed CRB expressions for sparse arrays explain
  - why more sources than sensors ($D \geq |\mathbb{S}|$) can be identified, and
  - why the CRB stagnates for large SNR and $D \geq |\mathbb{S}|$.
- More details can be found in our journal and website.[1]
- Other recent papers: Koochakzadeh and Pal[2] and by Wang and Nehorai.[3]
- Future: study the optimal array geoemtry for CRB.
- Work supported by Office of Naval Research.

Thank you!

[1] Liu and Vaidyanathan, *Digital Signal Proc.,Elsevier*, 2016; `http://systems.caltech.edu/dsp/students/clliu/CRB.html`
[2] Koochakzadeh and Pal, *IEEE Signal Proc. Letter*, 2016.
[3] Wang and Nehorai, *arXiv:1605.03620 [stat.AP]*, 2016.