New Cramér-Rao Bound Expressions for Coprime and Other Sparse Arrays

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***Abstract*—The Cramér-Rao bound (CRB) offers a lower bound on the variances of unbiased estimates of parameters, e.g., directions of arrival (DOA) in array processing. While there exist landmark papers on the study of the CRB in the context of array processing, the closed-form expressions available in the literature are not easy to use in the context of sparse arrays (such as minimum redundancy arrays (MRAs), nested arrays, or coprime arrays) for which the number of identifiable sources D exceeds the number of sensors N. Under such situations, the existing literature does not spell out the conditions under which the Fisher information matrix is nonsingular, or the condition under which specific closed-form expressions for the CRB remain valid. This paper derives a new expression for the CRB to fill this gap. The conditions for validity of this expression are expressed as the rank condition of a matrix defined based on the difference coarray. The rank condition and the closed-form expression lead to a number of new insights. For example, it is possible to prove the previously known experimental observation that, when there are more sources than sensors, the CRB stagnates to a constant value as the SNR tends to infinity.[1]**

***Index Terms*—Cramér-Rao bounds, Fisher information matrices, Coprime arrays, Sparse arrays, Difference coarray.**

I. Introduction

The Cramér-Rao bound (CRB), which offers a lower bound on the variances of unbiased estimates of the parameters, has found significant use in direction-of-arrival (DOA) problems [1]–[4]. Closed-form expressions for the CRB offer insights into the dependence of the array performance with respect to various parameters such as the number of sensors N in the array, the array geometry, the number of sources D, the number of snapshots, signal to noise ratio (SNR), and so forth.

The reason for the renewed interest in finding more useful closed-form expressions for the CRB is the following. For a long time, **sparse arrays**, such as the minimum redundancy arrays (MRAs) have been known to be able to identify more sources than sensors ($D \geq N$) [5]. More recently, the development of sparse arrays such as the nested arrays [6] and the coprime arrays [7], [8], have generated a new wave of interest in this topic. These new arrays have simple closed-form expressions for array geometry (compared to MRAs which do not have this advantage), which makes them more practical than MRAs. The most essential property of these successful sparse arrays is that, given N sensors, the difference coarrays of these arrays have $O(N^2)$ elements, which allows them to identify $D = O(N^2)$ sources using N sensors. In particular, therefore, $D \gg N$ is possible as demonstrated amply in [5]–[11].

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It is therefore of great importance to study the performance limits of these sparse arrays by using standard tools such as the CRB. If we try to do this using the existing results in the literature, we run into a road block. Either the known closed-form expressions are not valid when $D \geq N$ [1], or the precise conditions under which they are valid are not specified [3]. In this context, it is worth mentioning that the pioneering work by Abramovich *et al.* many years ago [10] discussed the performances of MRAs by successfully plotting the CRB even for the case of $D \geq N$. The same can be done today for nested and coprime arrays. However, the theoretical conditions under which the CRB exists (for the case $D \geq N$) have not been spelt out in the past.

We now summarize the main contributions of our paper. Starting from the Fisher information matrix for the case of stochastic CRB with uncorrelated priors, as in [3], we derive a new closed-form expression for the CRB, specifically for the case of uncorrelated sources. The new CRB expressions are valid if and only if the FIM is nonsingular. The condition for the validity of our CRB expression are here expressed explicitly in terms of the **augmented coarray manifold** matrix or the **ACM** matrix, which depends on the DOAs and the difference coarray. The main result is that the FIM is nonsingular if and only if the ACM matrix has full column rank, which is named as the **rank condition**. To the best of our knowledge, the invertibility of the FIM has not in the past been characterized in terms of the difference coarray geometry. The proposed CRB expression holds under this rank condition, and is given by our Eq. (14). Thus the specific CRB expression is valid whenever the FIM is invertible.

The invertibility of FIM, expressed as a rank condition on the ACM matrix, leads to a number of further insights as we shall elaborate in the paper. In short, the rank condition depends explicitly only on the difference coarray and the DOAs, whereas the CRB itself depends also on the physical array, the number of snapshots, and the SNR. We will also see that if the rank condition on the ACM matrix is satisfied, then the CRB converges to zero as the number of snapshots increases. However, the CRB stagnates to a constant value as the SNR goes to infinity when the array manifold matrix has full row rank. Similar behavior for $D \geq N$ was first noticed by Abramovich *et al.* in [10] experimentally. Here we elaborate the conditions and find these to be provable consequences of the specific CRB expression we derive. In particular, based on the results in [12], for nested arrays, coprime arrays, and MRAs, the FIM is provably invertible for $O(N^2)$ uncorrelated sources (the exact number depending on the specific array used, the source locations, and so forth), and therefore the CRB expression is provably valid for this many sources.

Notation. Scalars, vectors, matrices, and sets are denoted by lower-case letters (a), lower-case letters in bold face ($\mathbf{a}$), upper-case letters in bold face ($\mathbf{A}$), and upper-case letters in blackboard boldface ($\mathbb{A}$). $[\mathbf{A}]_{i,j}$ indicates the (i,j)th entry of $\mathbf{A}$. For a vector $\mathbf{x}_\mathbb{S}$ defined on $\mathbb{S}$ and $n \in \mathbb{S}$, $\langle \mathbf{x}_\mathbb{S} \rangle_n$ is the triangular bracket notation [11]. The complex conjugate, the transpose, and the complex conjugate transpose of $\mathbf{A}$ are $\mathbf{A}^*$, $\mathbf{A}^T$, and $\mathbf{A}^H$, respectively. The notation $\otimes$ is the Kronecker product. For a full column rank matrix $\mathbf{A}$, the matrix

$$\mathbf{\Pi}_\mathbf{A}^\perp = \mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H, \tag{1}$$

denotes the orthogonal projection onto the null space of $\mathbf{A}^H$. $\mathrm{diag}(a_1, \ldots, a_n)$ is a diagonal matrix with diagonals $a_1, \ldots, a_n$. For a real set $\mathbb{A} = \{a_1, \ldots, a_n\}$ such that $a_1 < \cdots < a_n$, $\mathrm{diag}(\mathbb{A}) = \mathrm{diag}(a_1, \ldots, a_n)$. $\mathrm{rank}(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A})$ denote the rank and the trace of $\mathbf{A}$, respectively. $\mathrm{vec}(\cdot)$ is the vectorization operation. The cardinality of a set $\mathbb{A}$ is $|\mathbb{A}|$. $\mathbb{E}[\cdot]$ denotes the expectation operator. $\mathcal{CN}(\mathbf{m}, \mathbf{\Sigma})$ is a complex normal distribution with mean $\mathbf{m}$ and covariance matrix $\mathbf{\Sigma}$.

II. Preliminaries

In sensor array processing, the sensor locations nd are described by an integer set $\mathbb{S}$ such that $n \in \mathbb{S}$, and $d = \lambda/2$ is half of the wavelength. We assume that this sensor array $\mathbb{S}$ is illuminated by D monochromatic plane waves with DOA θ_i satisfying $-\pi/2 \leq \theta_i \leq \pi/2$ for $i = 1, 2, \ldots, D$. Then, the measurements on the sensor array $\mathbb{S}$ can be modeled as [4]

$$\mathbf{x}_\mathbb{S} = \sum_{i=1}^{D} A_i \mathbf{v}_\mathbb{S}(\bar{\theta}_i) + \mathbf{n}_\mathbb{S} \quad \in \mathbb{C}^{|\mathbb{S}|}, \tag{2}$$

where A_i and $\bar{\theta}_i = (d/\lambda)\sin\theta_i$ represent the complex amplitude and the normalized DOA of the ith source. $\mathbf{n}_\mathbb{S}$ is a random noise term. $\mathbf{v}_\mathbb{S}(\bar{\theta}_i) = [e^{j2\pi\bar{\theta}_i n}]_{n\in\mathbb{S}}$ is the steering vector on $\mathbb{S}$.

In this paper, it is assumed that the uncorrelated sources have complex normal distribution. The power of the ith source is $p_i > 0$. The noise vector $\mathbf{n}_\mathbb{S}$ satisfies $\mathbf{n}_\mathbb{S} \sim \mathcal{CN}(\mathbf{0}, p_n\mathbf{I})$, where $p_n > 0$ is the noise power. Furthermore, sources are uncorrelated to noise, namely, $\mathbb{E}[A_i \mathbf{n}_\mathbb{S}^H] = \mathbf{0}$. Under these assumptions, $\mathbf{x}_\mathbb{S} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_\mathbb{S})$, where

$$\mathbf{R}_\mathbb{S} = \mathbb{E}[\mathbf{x}_\mathbb{S}\mathbf{x}_\mathbb{S}^H] = \sum_{i=1}^{D} p_i \mathbf{v}_\mathbb{S}(\bar{\theta}_i)\mathbf{v}_\mathbb{S}^H(\bar{\theta}_i) + p_n\mathbf{I}. \tag{3}$$

Vectorizing (3) and removing duplicated entries give [6], [13]

$$\mathbf{x}_\mathbb{D} = \sum_{i=1}^{D} p_i \mathbf{v}_\mathbb{D}(\bar{\theta}_i) + p_n\mathbf{e}_0, \tag{4}$$

where $\mathbf{v}_\mathbb{D}(\bar{\theta}_i)$ are the steering vectors on the difference coarray $\mathbb{D}$ and $\mathbf{e}_0$ is a column vector satisfying $\langle \mathbf{e}_0 \rangle_m = \delta_{m,0}$. Here $\mathbf{x}_\mathbb{D}$ can be regarded as a *deterministic* data vector on the difference coarray $\mathbb{D}$, which is defined as

Definition 1 (Difference coarray $\mathbb{D}$)**.** *Let $\mathbb{S}$ be an integer set defining the sensor locations. The difference set is defined as $\mathbb{D} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{S}\}$.*

The finite-snapshot version of (3) and (4) facilitates a variety of DOA estimators. Assume that $\widetilde{\mathbf{x}}_\mathbb{S}(k)$ for $k = 1, 2, \ldots, K$ denote K snapshots of $\mathbf{x}_\mathbb{S}$. The sample covariance matrix is $\widetilde{\mathbf{R}}_\mathbb{S} = \sum_{k=1}^{K} \widetilde{\mathbf{x}}_\mathbb{S}(k)\widetilde{\mathbf{x}}_\mathbb{S}^H(k)/K$. The finite-snapshot version of $\mathbf{x}_\mathbb{D}$, denoted by $\widetilde{\mathbf{x}}_\mathbb{D}$, can be constructed from $\widetilde{\mathbf{R}}_\mathbb{S}$ [6], [11]. Then, the augmented covariance matrix method [9] constructs an augmented covariance matrix from $\widetilde{\mathbf{R}}_\mathbb{S}$. The spatially smoothed MUSIC algorithm (SS MUSIC) [6], [8] evaluates a spatially smoothed matrix based on $\widetilde{\mathbf{x}}_\mathbb{D}$. These methods can resolve $O(N^2)$ uncorrelated sources for suitable array geometries like MRAs and coprime arrays with N sensors.

III. New Cramér-Rao Bound Expressions

In this section, we will propose a new CRB expression that remains valid for sparse arrays detecting $O(N^2)$ sources, such as nested arrays, coprimes arrays, and MRAs. First, it will be shown that a rank condition on the augmented coarray manifold (ACM) matrix is necessary and sufficient for the nonsingular FIM, which leads to a closed-from CRB expression. The detailed proofs of lemmas and theorems can be found in [12].

Consider a random vector $\mathbf{x}$ with a complex normal distribution with mean zero and covariance $\mathbf{\Sigma}(\boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is a real-valued parameter vector. Then, the (p,ℓ)th entry of the FIM $\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})$ is given by [14]

$$[\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})]_{p,\ell} = \mathrm{tr}\left(\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha})\frac{\partial\mathbf{\Sigma}(\boldsymbol{\alpha})}{\partial[\boldsymbol{\alpha}]_p}\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha})\frac{\partial\mathbf{\Sigma}(\boldsymbol{\alpha})}{\partial[\boldsymbol{\alpha}]_\ell}\right). \tag{5}$$

Taking $\boldsymbol{\alpha} = [\bar{\theta}_1, \ldots, \bar{\theta}_D, p_1, \ldots, p_D, p_n]^T$ and using (5) for K snapshots of $\mathbf{x}_\mathbb{S}$ lead to [3], [14]

$$[\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})]_{p,\ell} = K\mathrm{vec}^H\!\left(\frac{\partial\mathbf{R}_\mathbb{S}}{\partial[\boldsymbol{\alpha}]_p}\right)(\mathbf{R}_\mathbb{S}^{-T}\otimes\mathbf{R}_\mathbb{S}^{-1})\mathrm{vec}\!\left(\frac{\partial\mathbf{R}_\mathbb{S}}{\partial[\boldsymbol{\alpha}]_\ell}\right), \tag{6}$$

since $\mathrm{tr}(\mathbf{ABCD}) = \mathrm{vec}(\mathbf{B}^H)^H(\mathbf{A}^T\otimes\mathbf{C})\mathrm{vec}(\mathbf{D})$. (6) can be expressed as

$$\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha}) = K\begin{bmatrix}\mathbf{G} & \mathbf{\Delta}\end{bmatrix}^H\begin{bmatrix}\mathbf{G} & \mathbf{\Delta}\end{bmatrix}, \tag{7}$$

where

$$\mathbf{G} = \left(\mathbf{R}_\mathbb{S}^T\otimes\mathbf{R}_\mathbb{S}\right)^{-\frac{1}{2}}\begin{bmatrix}\frac{\partial\mathbf{r}_\mathbb{S}}{\partial\bar{\theta}_1} & \cdots & \frac{\partial\mathbf{r}_\mathbb{S}}{\partial\bar{\theta}_D}\end{bmatrix}, \tag{8}$$

$$\mathbf{\Delta} = \left(\mathbf{R}_\mathbb{S}^T\otimes\mathbf{R}_\mathbb{S}\right)^{-\frac{1}{2}}\begin{bmatrix}\frac{\partial\mathbf{r}_\mathbb{S}}{\partial p_1} & \cdots & \frac{\partial\mathbf{r}_\mathbb{S}}{\partial p_D} & \frac{\partial\mathbf{r}_\mathbb{S}}{\partial p_n}\end{bmatrix}, \tag{9}$$

and $\mathbf{r}_\mathbb{S} = \mathrm{vec}(\mathbf{R}_\mathbb{S})$. It follows from (7) that the FIM is positive semidefinite. And $\mathbf{\Delta}^H\mathbf{\Delta}$ is obviously positive semidefinite. If the FIM is nonsingular, then the CRB for the normalized DOAs $\bar{\boldsymbol{\theta}} = [\bar{\theta}_1, \ldots, \bar{\theta}_D]^T$ can be expressed as the inverse of the Schur complement of the block $\mathbf{\Delta}^H\mathbf{\Delta}$ of $\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})$ [3], [14]

$$\mathrm{CRB}(\bar{\boldsymbol{\theta}}) = \frac{1}{K}\left(\mathbf{G}^H\mathbf{\Pi}_\mathbf{\Delta}^\perp\mathbf{G}\right)^{-1}, \tag{10}$$

where $\mathbf{\Pi}_\mathbf{\Delta}^\perp$ is defined as in (1). Notice that this CRB expression (10) is valid *if and only if* $\mathbf{\Delta}^H\mathbf{\Delta}$ and $\mathbf{G}^H\mathbf{\Pi}_\mathbf{\Delta}^\perp\mathbf{G}$ are both nonsingular. It is of great interest to simplify the condition that $\mathbf{\Delta}^H\mathbf{\Delta}$ and $\mathbf{G}^H\mathbf{\Pi}_\mathbf{\Delta}^\perp\mathbf{G}$ are both nonsingular. To proceed, we first define the ACM matrix $\mathbf{A}_c$ as follows:

Definition 2 (ACM matrix)**.** *The augmented coarray manifold (ACM) matrix is defined as*

$$\mathbf{A}_c = \begin{bmatrix}\operatorname{diag}(\mathbb{D})\mathbf{V}_{\mathbb{D}} & \mathbf{W}_{\mathbb{D}}\end{bmatrix}, \tag{11}$$

where $\mathbf{V}_{\mathbb{D}}$*,* $\mathbf{W}_{\mathbb{D}}$ *are given by*

$$\mathbf{V}_{\mathbb{D}} = \begin{bmatrix}\mathbf{v}_{\mathbb{D}}(\bar{\theta}_1) & \mathbf{v}_{\mathbb{D}}(\bar{\theta}_2) & \dots & \mathbf{v}_{\mathbb{D}}(\bar{\theta}_D)\end{bmatrix}, \tag{12}$$

$$\mathbf{W}_{\mathbb{D}} = \begin{bmatrix}\mathbf{V}_{\mathbb{D}} & \mathbf{e}_0\end{bmatrix}. \tag{13}$$

Here $\mathbf{e}_0$ *is a column vector satisfying* $\langle \mathbf{e}_0\rangle_m = \delta_{m,0}$ *and* $\mathbb{D}$ *is the difference coarray, as given in Definition 1.*

The following lemmas [12] characterize the necessary and sufficient conditions that $\mathbf{\Delta}^H\mathbf{\Delta}$ and $\mathbf{G}^H\mathbf{\Pi}^{\perp}_{\mathbf{\Delta}}\mathbf{G}$ are positive definite, hence nonsingular.

Lemma 1. $\mathbf{\Delta}^H\mathbf{\Delta}$ *is positive definite if and only if* $\mathbf{W}_{\mathbb{D}}$ *has full column rank, i.e., if and only if* $\operatorname{rank}(\mathbf{W}_{\mathbb{D}}) = D+1$.

Lemma 2. *Assume that* $\operatorname{rank}(\mathbf{W}_{\mathbb{D}}) = D+1$*. Then* $\mathbf{G}^H\mathbf{\Pi}^{\perp}_{\mathbf{\Delta}}\mathbf{G}$ *is positive definite if and only if the ACM matrix* $\mathbf{A}_c$ *has full column rank, i.e., if and only if* $\operatorname{rank}(\mathbf{A}_c) = 2D+1$.

The significance of Lemma 1 and Lemma 2 is that the invertibility of $\mathbf{\Delta}^H\mathbf{\Delta}$ and $\mathbf{G}^H\mathbf{\Pi}^{\perp}_{\mathbf{\Delta}}\mathbf{G}$ can be simply characterized by the rank of the ACM matrix. Furthermore, these conditions lead to a necessary and sufficient condition for nonsingular FIMs, as summarized next [12]:

Theorem 1. *The FIM* $\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})$ *is nonsingular if and only if* $\mathbf{A}_c$ *has full column rank, i.e., if and only if* $\operatorname{rank}(\mathbf{A}_c) = 2D+1$.

We will denote the condition that $\mathbf{A}_c$ has full column rank as the *rank condition*. The next result is that, if the FIM is nonsingular, then the CRB exists and the closed-form CRB expression is given by the following theorem [12]:

Theorem 2. *If* $\mathbf{A}_c$ *has full column rank, then the CRB for normalized DOAs* $\bar{\boldsymbol{\theta}} = [\bar{\theta}_1, \dots, \bar{\theta}_D]^T$ *can be expressed as*

$$\operatorname{CRB}(\bar{\boldsymbol{\theta}}) = \frac{1}{4\pi^2 K}\left(\mathbf{G}_0^H \mathbf{\Pi}^{\perp}_{\mathbf{M}\mathbf{W}_{\mathbb{D}}}\mathbf{G}_0\right)^{-1}, \tag{14}$$

where $\mathbf{G}_0 = \mathbf{M}(\operatorname{diag}(\mathbb{D}))\mathbf{V}_{\mathbb{D}}(\operatorname{diag}(p_1, p_2, \dots, p_D))$ *and* $\mathbf{M} = (\mathbf{J}^H(\mathbf{R}_{\mathbb{S}}^T \otimes \mathbf{R}_{\mathbb{S}})^{-1}\mathbf{J})^{1/2}$*. The binary matrix* $\mathbf{J}$ *has size* $|\mathbb{S}|^2$*-by-*$|\mathbb{D}|$ *such that the column of* $\mathbf{J}$ *associated with the difference* m *is given by* $\langle \mathbf{J}\rangle_{:,m} = \operatorname{vec}(\mathbf{I}(m))$ *for* $m \in \mathbb{D}$*. The* $|\mathbb{S}|$*-by-*$|\mathbb{S}|$ *matrix* $\mathbf{I}(m)$ *satisfies*

$$\langle \mathbf{I}(m)\rangle_{n_1,n_2} = \begin{cases}1 & \text{if } n_1 - n_2 = m,\\ 0 & \text{otherwise,}\end{cases} \qquad n_1, n_2 \in \mathbb{S}.$$

Theorem 2 enables us to study the parameters that affect the CRB, such as the array configuration, the normalized DOAs, the number of snapshots, and the SNR, as explained next.

Property 1. *The rank condition depends only on four factors: the difference coarray* $\mathbb{D}$*, the normalized DOAs* $\bar{\boldsymbol{\theta}}$*, the number of sources* D*, and* $\mathbf{e}_0$*. The following parameters are irrelevant to the rank condition: The source powers* $p_1, \dots, p_D > 0$*, the noise power* $p_n > 0$*, and the number of snapshots* K*.*

Property 2. *The CRB for* $\bar{\boldsymbol{\theta}}$ *is a function of the physical array* $\mathbb{S}$*, the normalized DOA* $\bar{\boldsymbol{\theta}}$*, the number of sources* D*, the number of snapshots* K*, and the SNR of sources* $p_1/p_n, \dots, p_D/p_n$*.*

Property 3. *If* $\operatorname{rank}(\mathbf{A}_c) = 2D+1$*, then as the number of snapshots* K *approaches infinity,* $\operatorname{CRB}(\bar{\boldsymbol{\theta}})$ *converges to zero.*

The following theorems investigate the asymptotic behavior of the CRB for large SNR. Assume the sources have identical power. It was experimentally noticed in [10] that for $D < |\mathbb{S}|$, the CRB decays to zero for large SNR while for $D \geq |\mathbb{S}|$, the CRB tends to converge to a non-zero value for large SNR. Here we find these phenomena to be a provable consequence of the proposed CRB expression as given in Theorem 2.

However, in this paper, we notice that the conditions $D < |\mathbb{S}|$ and $D \geq |\mathbb{S}|$ are not fundamental to the asymptotic behavior of the CRB for large SNR. Instead, the condition that the array manifold matrix $\mathbf{V}_{\mathbb{S}}$ has full row rank, i.e., $\operatorname{rank}(\mathbf{V}_{\mathbb{S}}) = |\mathbb{S}|$, is more critical. In the regime $D < |\mathbb{S}|$, $\mathbf{V}_{\mathbb{S}}$ does not have full row rank since $\mathbf{V}_{\mathbb{S}}$ is a tall matrix. Thus, the asymptotic CRB expression can be specified by the following theorem [12]:

Theorem 3. *If the* D *uncorrelated sources have equal SNR* p/p_n*,* $\operatorname{rank}(\mathbf{V}_{\mathbb{S}}) < |\mathbb{S}|$*, and* $\operatorname{rank}(\mathbf{A}_c) = 2D+1$*, then for sufficiently large SNR, the CRB has the following asymptotic expression which converges to zero as SNR tends to infinity:*

$$\operatorname{CRB}(\bar{\boldsymbol{\theta}})\Big|_{\substack{\text{large SNR}\\ \operatorname{rank}(\mathbf{V}_{\mathbb{S}})<|\mathbb{S}|}} = \frac{p_n}{4\pi^2 K p}\mathbf{S}^{-1}, \tag{15}$$

where

$$\mathbf{S} = \mathbf{G}_{\infty}^H \mathbf{\Pi}^{\perp}_{\mathbf{M}_{\infty}\mathbf{W}_{\mathbb{D}}}\mathbf{G}_{\infty} + (\mathbf{G}_{\infty}^H\mathbf{u})(\mathbf{G}_{\infty}^H\mathbf{u})^H / \|\mathbf{u}\|^2, \tag{16}$$

$$\mathbf{M}_{\infty} = \left[\mathbf{J}^H\left[(\mathbf{U}_s\mathbf{\Lambda}^{-1}\mathbf{U}_s^H)^T \otimes (\mathbf{U}_n\mathbf{U}_n^H) + (\mathbf{U}_n\mathbf{U}_n^H)^T \otimes (\mathbf{U}_s\mathbf{\Lambda}^{-1}\mathbf{U}_s^H)\right]\mathbf{J}\right]^{\frac{1}{2}}, \tag{17}$$

$$\mathbf{u} = (\mathbf{M}_{\infty}\mathbf{W}_{\mathbb{D}})\left(\mathbf{W}_{\mathbb{D}}^H\mathbf{M}_{\infty}^2\mathbf{W}_{\mathbb{D}}\right)^{-1}\mathbf{e}_{D+1}, \tag{18}$$

$\mathbf{G}_{\infty} = \mathbf{M}_{\infty}(\operatorname{diag}(\mathbb{D}))\mathbf{V}_{\mathbb{D}}$*, and the* $(D+1)$*-dimensional vector* $\mathbf{e}_{D+1} = [0, \dots, 0, 1]^T$*. Here* $\mathbf{V}_{\mathbb{S}}\mathbf{V}_{\mathbb{S}}^H$ *has eigen-decomposition* $\mathbf{U}_s\mathbf{\Lambda}\mathbf{U}_s^H$*.* $\mathbf{U}_n$ *is orthonormal to* $\mathbf{U}_s$*.* $\mathbf{W}_{\mathbb{D}}^H\mathbf{M}_{\infty}^2\mathbf{W}_{\mathbb{D}}$ *and* $\mathbf{S}$ *can be readily shown to be positive definite.* $\mathbf{u}$ *can be shown to be non-zero.*

It is obvious from (15) that, as the SNR approaches infinity, the CRB decays to zero for $D < |\mathbb{S}|$, which is consistent with the observation in [10].

For $D \geq |\mathbb{S}|$ and $\mathbf{V}_{\mathbb{S}}$ being full row rank, the asymptotic CRB expression can be given by [12]

Theorem 4. *If the* D *uncorrelated sources have equal SNR* p/p_n*,* $D \geq |\mathbb{S}|$*,* $\operatorname{rank}(\mathbf{V}_{\mathbb{S}}) = |\mathbb{S}|$*, and* $\operatorname{rank}(\mathbf{A}_c) = 2D+1$*, then for sufficiently large SNR, the CRB has an asymptotic expression which does not decay to zero as SNR tends to infinity. Thus,*

$$\operatorname{CRB}(\bar{\boldsymbol{\theta}})\Big|_{\substack{\text{large SNR}\\ \operatorname{rank}(\mathbf{V}_{\mathbb{S}})=|\mathbb{S}|}} = \frac{1}{4\pi^2 K}\mathbf{S}^{-1}, \tag{19}$$

where $\mathbf{S} = \mathbf{G}_{\infty}^H\mathbf{\Pi}^{\perp}_{\mathbf{M}_{\infty}\mathbf{W}_{\mathbb{D}}}\mathbf{G}_{\infty}, \mathbf{M}_{\infty} = (\mathbf{J}^H((\mathbf{V}_{\mathbb{S}}\mathbf{V}_{\mathbb{S}}^H)^{-T} \otimes (\mathbf{V}_{\mathbb{S}}\mathbf{V}_{\mathbb{S}}^H)^{-1})\mathbf{J})^{\frac{1}{2}}, \mathbf{G}_{\infty} = \mathbf{M}_{\infty}(\operatorname{diag}(\mathbb{D}))\mathbf{V}_{\mathbb{D}}$*. It can be shown that* $\mathbf{W}_{\mathbb{D}}^H\mathbf{M}_{\infty}^2\mathbf{W}_{\mathbb{D}}$ *and* $\mathbf{S}$ *are positive definite.*

Theorem 4 also confirms what was empirically observed in [10], for $D \geq |\mathbb{S}|$. It will be demonstrated in Section IV that the proposed CRB expression (14) indeed comes close to the asymptotic values (15) and (19).

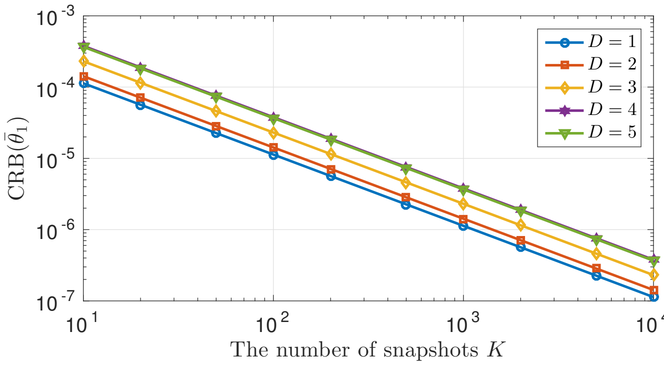

Fig. 1. The dependence of the proposed CRB expression on snapshots for various numbers of sources D. The array configuration is the nested array with $N_1 = N_2 = 2$ so that the sensor locations are $\mathbb{S} = \{1, 2, 3, 6\}$. The equal-power sources are located at $\bar{\theta}_i = -0.49 + 0.95(i-1)/D$ for $i = 1, 2, \ldots, D$. The SNR is 0 dB.

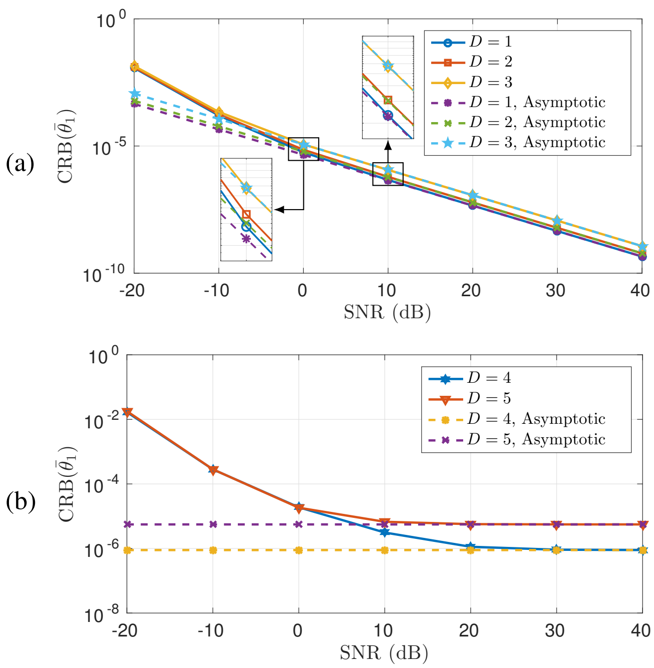

Fig. 2. The dependence of the proposed CRB expression on SNR for (a) $D < |\mathbb{S}| = 4$ and (b) $D \geq |\mathbb{S}| = 4$. The array configuration is the nested array with $N_1 = N_2 = 2$ so that the sensor locations are $\mathbb{S} = \{1, 2, 3, 6\}$. The equal-power sources are located at $\bar{\theta}_i = -0.49 + 0.95(i-1)/D$ for $i = 1, 2, \ldots, D$. The number of snapshots K is 200.

IV. Numerical Examples

Our first numerical example examines Property 3, Theorem 3, and Theorem 4. Consider a nested array with $N_1 = N_2 = 2$, so that the sensor locations $\mathbb{S} = \{1, 2, 3, 6\}$ and the difference coarray becomes $\mathbb{D} = \{-5, \ldots, 5\}$ [6]. As a result, the total number of sensors is 4 while the maximum number of identifiable sources is 5. The equal-power sources are located at $\bar{\theta}_i = -0.49 + 0.95(i-1)/D$ for $i = 1, 2, \ldots, D$. It can be shown that these parameters indeed satisfy the rank condition so that the proposed CRB expression is valid.

Fig. 1 plots the proposed CRB expression for $\bar{\theta}_1$ as a function of snapshots, with 0 dB SNR. It can be observed

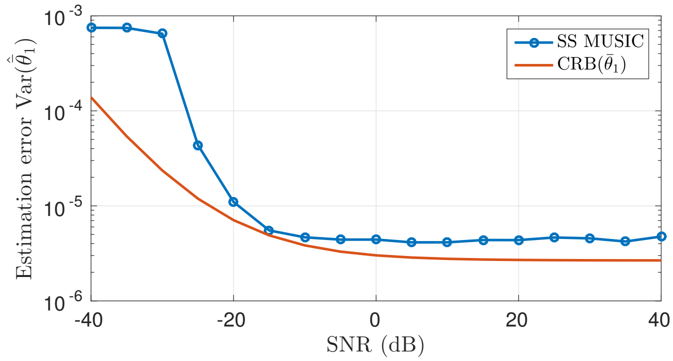

Fig. 3. The error variance for SS MUSIC and the proposed CRB expression. The array geometry is the coprime array with $M = 3, N = 5$ so at most 17 uncorrelated sources can be identified. The equal-power sources are located at $\bar{\theta}_i = -0.49 + 0.95(i-1)/D$ for $i = 1, 2, \ldots, D$. The number of sources $D = 17$, which is greater than the number of sensors 10. The number of snapshots K is 500. Each point is an average over 1000 Monte-Carlo runs.

that this expression is inversely proportional to the number of snapshots K, which verifies Property 3. These curves also depend on the number of sources D. In this specific example, these CRBs increase with D, which suggests that if there are more sources, it is more difficult to estimate $\bar{\theta}_1$ accurately.

Fig. 2(a) and (b) display the relationship between the proposed CRB expression and the SNR for 200 snapshots. Fig. 2(a) shows that if $D < |\mathbb{S}| = 4$, the CRBs decrease with the SNR. For $D \geq |\mathbb{S}| = 4$, the CRBs saturate when the SNR is over 20dB, as indicated in Fig. 2(b). These phenomena are consistent with what was observed experimentally in [10]. Furthermore, the dashed lines in Fig. 2(a) and (b) demonstrate that, for large SNR, the CRBs indeed converge to the asymptotic CRB expressions, as presented in Theorem 3 and 4.

The next example compares the estimation performance for SS MUSIC with the proposed CRB expression. Consider an example of the coprime array with $M = 3, N = 5$ so the sensor locations are $0, 3, 5, 6, 9, 10, 12, 15, 20, 25$ and the difference coarray contains consecutive integers from -17 to 17 [8]. Therefore, up to 17 uncorrelated sources can be identified. Fig. 3 shows the dependence of estimation performance for $\bar{\theta}_1$ on SNR, with 1000 runs, 500 snapshots and $D = 17$ uncorrelated sources. It can be deduced that, first, the proposed CRB expression is a lower bound of the estimation variance for SS MUSIC. Furthermore, the SS MUSIC does not achieve the CRB, even when the SNR is as large as 40 dB.

V. Concluding Remarks

In this paper, we derived a new expression for the CRB of DOA estimates. This expression is useful for sparse arrays such as nested arrays, coprime arrays, or MRAs, which can identify more sources than sensors. The conditions for validity of the expression are expressed in terms of the rank of an augmented coarray manifold matrix. Considerable insights regarding the behavior of sparse arrays can be gained from these expressions. In the future, it will be of interest to study the dependence of the CRB on the array configurations.

During the final revision of this paper we came to know that somewhat similar results are being reported by Koochakzadeh and Pal [15] and by Wang and Nehorai [16].

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